Revision answers: Algebra, functions and equations (Topics 1 & 2)

Coursebook chapters: 1–8; 15; 25

1. (a) $\mathbb{R}$

(b) $f^{-1}(x) = \frac{1}{3}e^{2x}$

(c) $f^{-1}(5) = \frac{1}{3}e^{10}$ *[5 marks]*

2. $a - bi + 2a + 2bi = 3i \Rightarrow a = 0, b = 3$ *[4 marks]*

3. $8a - b = -13, \frac{a}{8} + b = \frac{9}{4} \quad \therefore\ a = -\frac{86}{65}, b = \frac{157}{65}$ *[5 marks]*

4. (a) $y > 1$

(b) $3(4e^x + 1)^2 = 75 \Rightarrow 4e^x + 1 = 5 \Rightarrow x = 0$ *[8 marks]*

5. (a) $\binom{2}{1} \times \binom{6}{3} = 40$

(b) $8! - 2 \times 7! = 30240$ *[6 marks]*

6. (a) Use GDC to sketch $y = 2(x - 1)^2 - 6$

(b) From GDC: $-1.35 < x < 3.35$ *[5 marks]*

7. (a) Arithmetic series, $u_1 = 500$, $d = 25$, $S_{20} = 14750$

(b) Geometric, $u_1 = 500$, $r = 1.05$, $u_n = 500 \times 1.05^{n-1} > 1000 \Rightarrow n = 16$ (from GDC)

(c) $\frac{500(1.05^n - 1)}{0.05} > \frac{n}{2}(1000 + 25(n - 1)) = 5000$ gives $n = 27$ days *[9 marks]*

8. (a) $x = \frac{b}{2}$

(b) $f(x) = f^{-1}$ is equivalent to $f(x) = x$ (because the two graphs cross on the line $y = x$.

$\frac{x+3}{2x-5} = x \Rightarrow x = -0.436$ or 3.44 *[5 marks]*

9. (a) $p(3) = 0$

(b) $p(x) = (x - 3)(2x^2 + x - 3) = (x - 3)(2x + 3)(x - 1)$

(c) Cubic graph with x-intercepts -1.5, 1, 3 and y-intercept 9. *[9 marks]*

10. The inductive step is:

For $n = k$, $5^k + 9^k + 2 = 4A$, $n = k$,

so for $n = k + 1$, $5^{k+1} + 9^{k+1} + 2 = 5 \times 5^k + 9 \times (4A - 5^k - 2) + 2 = 36A - 4 \times 5^k - 16$

$= 4(9A - 5^k - 4)$ $\therefore$ divisible by 4

[9 marks]

11. Gaussian elimination gives:

$$\begin{cases} x - 2y + z = 5 \\ 5y = 1 \\ 0 = 0 \end{cases}$$

So, $z = t$, $y = 0.2$, $x = 5.4 - t$ *[7 marks]*

12. (a) The inductive step uses the compound-angle formulae for sine and cosine. (See Worked example 15.19 for a similar method.)

(b) (i) $\omega = e^{\frac{2\pi}{5}i}$

(ii) $1 + \omega + \omega^2 + \omega^3 + \omega^4 = \frac{1-\omega^5}{1-\omega} = 0$ as $\omega^5 = 1$ *[10 marks]*

13. $x_1 + x_2 = -\frac{b}{a}$, $x_1 - x_2 = 1 \;\Rightarrow\; (x_2 + 1) + x_2 = -\frac{b}{a} \;\Rightarrow\; x_2 = \frac{1}{2}\left(-\frac{b}{a} - 1\right) = -\frac{b}{2a} - \frac{1}{2}$

$$x_1 x_2 = \frac{c}{a} \;\Rightarrow\; \left(-\frac{b}{2a} + \frac{1}{2}\right)\left(-\frac{b}{2a} - \frac{1}{2}\right) = \frac{c}{a}$$

$$\Rightarrow \left(\frac{b}{2a}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{c}{a} \;\Rightarrow\; \frac{b^2 - a^2}{4a^2} = \frac{c}{a} \;\Rightarrow\; b^2 - a^2 = 4ac$$

[6 marks]

14. $x^n + npx^{n-1} + \frac{n(n-1)}{2}p^2 x^{n-2} = x^n + 20x^{n-1} + 180x^{n-2}$

$$\Rightarrow\; np = 20,\; \frac{n(n-1)}{2}p^2 = 180$$

$$\Rightarrow\; \frac{n(n-1)}{2}\frac{400}{n^2} = 180 \;\Rightarrow\; n = 10,\, p = 2$$

[6 marks]

15. $p(x) = (x^2 + 3x + 2)q(x) + (5x + 1)$

$p(-2) = (0)q(x) + (-10 + 1) = -9$

[4 marks]

16. (a) $\alpha + \beta + \gamma = -b$ and $\gamma = \alpha + \beta \;\Rightarrow\; 2(\alpha + \beta) = -b$

$$\alpha\beta\gamma = -d \;\Rightarrow\; \alpha\beta = -\frac{d}{\alpha + \beta} = \frac{2d}{b}$$

(b) If α and β are solutions of the quadratic equation then $x^2 + mx + n = (x - \alpha)(x - \beta)$.

Therefore $\alpha\beta = n$ and $-(\alpha + \beta) = m$, which gives:

$$m = \frac{b}{2},\; n = \frac{2d}{b}$$

(c) Three real roots when α, β are real, i.e. $m^2 - 4n \geq 0 \;\Leftrightarrow\; \frac{b^2}{4} \geq \frac{8d}{b} \;\Leftrightarrow\; b^3 \geq 32d$ (as $b > 0$).

[9 marks]