Extension worksheet 1 Counting principles

Questions

1. A necklace is to be made by threading four identical black beads and four identical white beads onto a string which is closed into a loop.
   How many different patterns can be made?
2. Given an unlimited supply of 50p, £1 and £2 coins, in how many different ways is it possible to make a sum of £100?
3. A tennis competition has 1093 entrants. In each round, if there is an even number of people, each entrant plays one other entrant and the winner progresses to the next round. If there is an odd number of people, one person gets a 'bye' (an entry to the next round, without playing a match) and the remaining entrants all play matches. So, in the first round, 546 matches are played and 1 bye is given. How many matches are played in total?

> There is a nice way of getting to this answer described in the answers. Is there a concept of beauty when judging mathematical solutions?

4. There are six addressed letters and six addressed envelopes. One letter is put into each envelope.
   (a) In how many ways can exactly two letters be put into incorrect envelopes?
   (b) In how many ways can exactly five letters be put into incorrect envelopes?
   (c) In how many ways can every letter be put into an incorrect envelope?

> This type of problem is called a derangement. There is an interesting link between derangements and number e.

5. An exam has six questions each marked out of 8. How many different sets of marks are there if:
   (a) the mark in each question is less than in the previous question?
   (b) the mark in each question is less than or equal to the previous question?

Extension worksheet 2 Exponents and logarithms

Questions

1. (a) Show that:

$$\sqrt{2+\sqrt{3}}-\sqrt{2-\sqrt{3}}=\sqrt{2}$$

(b) Hence prove that:

$$\log_{\frac{1}{8}}\left(\sqrt{2+\sqrt{3}}-\sqrt{2-\sqrt{3}}\right)=-\tfrac{1}{6}$$

(c) Find all possible pairs of integers a and n such that:

$$\log_{\frac{1}{n}}\left(\sqrt{a+\sqrt{15}}-\sqrt{a-\sqrt{15}}\right)=-\tfrac{1}{2}$$

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2. Given that $x > y > 0$,

(a) By writing $\log_y x = z$, or otherwise, show that $\log_y x = \dfrac{1}{\log_x y}$

(b) Given also that $\log_x y = \log_y x$, show that $y = \dfrac{1}{x}$

(c) Solve the simultaneous equations

$$\log_x y = \log_y x,$$
$$\log_x (x - y) = \log_y (x + y).$$

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3. (a) Anna, who is confused about the rules for logarithms, states that

$$(\log_3 p)^2 = \log_3 (p^2)$$

and $$\log_3 (p + q) = \log_3 p + \log_3 q.$$

However, there is a value for p and a value for q for which both statements are correct.

Find the value of p and the value of q.

(b) Solve $$\frac{\log_3(3x^3-23x^2+40x)}{\log_3 9} = 0.5 + \log_3 (3x - 8).$$

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4. (a) How many digits does 2^{500} have?

(b) What is the last digit of 2^{500}?

(c) What is the first digit of 2^{500}?

(d) What is the second digit of 2^{500}?

5. If $x=10^{100}$ and $y=100^{10}$ find, with proof, which is larger: x^y or y^x?

Extension worksheet 3 Polynomials

Questions

1. (a) Find the values of b for which the quadratic function $f(x) = x^2 - bx + 1$ has real roots.
   (b) Show that when $b > 2$, $f(x)$ has two positive roots.
   (c) When does $f(x)$ have two negative roots?
   (d) Fill in the table showing the number and the sign of the roots of $f(x)$ for various values of b:

b	*Number of real roots*	*Sign*
$b > 2$	two	positive
$b = 2$		
	none	
$b = -2$		
	two	

2. (a) Find the polynomial $f(x)$ of degree 10 such that $f(x) = 0$ for $x = 1, 2, \ldots, 10$ and $f(11) = 1$, and evaluate $f(12)$.
   (b) Let $g(x)$ be a polynomial of degree 10 such that $g(n) = n$ for $n = 1, 2, \ldots, 10$ and $g(11) = 12$. Find $g(12)$.

3. Let $f(n)$ be a polynomial of degree 1000. Suppose that for all integers $1 \le n \le 1001$, $f(n) = \frac{1}{n}$. Evaluate $f(1002)$.

4. $f(x)$ is a polynomial with integer coefficients. a, b, c and d are distinct integers such that $f(a) = f(b) = f(c) = f(d) = 5$. Prove that there is no integer value e such that $f(e) = 12$.

5. The quadratic equation $4x^2 - 4(k + 3)x + 5k + 8 = 0$, where k is a real number, has roots α and β.
   (a) Show that α and β are real and unequal for all values of k.
   (b) Given that k varies, find the least value of $|\alpha - \beta|$.
   (c) Given that $\alpha = 2\beta$, determine the values of k and β.

Extension worksheet 4 Algebraic structures

When we solve an equation algebraically we are saying that the final solution will work in the original equation. However, sometimes our working will introduce other solutions. For example, working with the equation $x = -1$:

$$
\begin{array}{ll}
 & x = -1 \\
\Rightarrow & x^2 = 1 \\
\Leftrightarrow & x = 1 \text{ or } x = -1
\end{array}
$$

Clearly $x = 1$ is not a solution to the original equation, so how has it been found in the working? The answer lies in the logical symbols on the left.

$\Leftrightarrow$ means 'this line is equivalent to'

$\Rightarrow$ means ' this line follows from the previous line'

$\Leftarrow$ means 'the previous line follows from this'

The third symbol means that the argument is not reversible: If $x = -1$ then we know that $x^2 = 1$ but if $x^2 = 1$ we cannot conclude that $x = -1$.

For the final answer to be a solution to the original equation all the implications must be 'equivalent to'. If there is a 'follows from' then we need to check that our solutions do work in the original equation.

Questions

1. Zhao was asked to solve the equation $2x = |x-1|$.

Here is her working:

$$
\begin{array}{ll}
 & 2x = |x-1| \\
\Leftrightarrow & (2x)^2 = |x-1|^2 \\
\Leftrightarrow & (2x)^2 = (x-1)^2 \\
\Leftrightarrow & 4x^2 = x^2 - 2x + 1 \\
\Leftrightarrow & 3x^2 + 2x - 1 = 0 \\
\Leftrightarrow & (3x-1)(x+1) = 0 \\
\Leftrightarrow & x = \frac{1}{3} \text{ or } x = -1
\end{array}
$$

(a) By checking her solutions, find the correct solution.

(b) In which line of working is her mistake?

2. Lambert was asked to solve the equation $x = \sqrt{(3x+4)}$.

Here is his working:

$$
\begin{aligned}
& & x &= \sqrt{(3x+4)} \\
&\Leftrightarrow & x^2 &= 3x+4 \\
&\Leftrightarrow & x^2 - 3x - 4 &= 0 \\
&\Leftrightarrow & (x-4)(x+1) &= 0 \\
&\Leftrightarrow & x &= 4 \text{ or } x = -1
\end{aligned}
$$

(a) By checking his solutions, find the correct solution.

(b) In which line of working is his mistake?

3. Craig was asked to solve the equation $x^2 = 3x$.

Here is his solution:

$$
\begin{aligned}
& & x^2 &= 3x \\
&\Leftrightarrow & x &= 3
\end{aligned}
$$

(a) Show that $x = 0$ is also a solution to the original equation.

(b) What logical symbol should Craig have used in the second line?

4. Freja was asked to solve the equation $x - \frac{1}{x-3} = 1 + \frac{5-2x}{x-3}$.

Here is her working:

$$
\begin{aligned}
& & x - \frac{1}{x-3} &= 1 + \frac{5-2x}{x-3} \\
&\Leftrightarrow & x - 1 &= \frac{6-2x}{x-3} \\
&\Leftrightarrow & (x-1)(x-3) &= 6-2x \\
&\Leftrightarrow & x^2 - 4x + 3 &= 6-2x \\
&\Leftrightarrow & x^2 - 2x - 3 &= 0 \\
&\Leftrightarrow & (x-3)(x+1) &= 0 \\
&\Leftrightarrow & x &= 3 \text{ or } x = -1
\end{aligned}
$$

(a) By checking her solutions, find the correct solution.

(b) In which line of working is her mistake?

5. Jamie was asked to solve $\log_2(-x)+\log_2(2-x)=3$.
Here is her working:

$$
\begin{aligned}
& & \log_2(-x)+\log_2(2-x) &= 3 \\
&\Leftrightarrow & \log_2\left(-x(2-x)\right) &= 3 \\
&\Leftrightarrow & \log_2(x^2-2x) &= 3 \\
&\Leftrightarrow & x^2-2x &= 2^3 \\
&\Leftrightarrow & x^2-2x-8 &= 0 \\
&\Leftrightarrow & (x-4)(x+2) &= 0 \\
&\Leftrightarrow & & x=4 \text{ or } x=-2
\end{aligned}
$$

(a) By checking her solutions, find the correct solution.

(b) In which line of working is her mistake?

Extension worksheet 5 The theory of functions

Questions

1. Sketch the following 'unusual' graphs for $1 \le x \le 8,\ x \in \mathbb{Z}$:

(a) $y =$ factors of x

(b) $y =$ distance to nearest square number

(c) $y =$ highest common factor of x and 12

(d) $y =$ number of letters in the English spelling of x.

> Some relations such as the one in (d) have remarkably interesting and stable mathematical properties. More can be found in the *On-line Encyclopaedia of Integer Sequences* (oeis.org).

2. Solve the following functional identities:

> A functional identity is a relationship that is true for all values of the variables in question. To solve a functional identity means to find all functions which satisfy it.

(a) $y\,f(x) = x\,f(y)$

(b) $f(xy) = x\,f(y)$

(c) $f(x+y) + f(x-y) = 2x^2 + 2y^2$

(d) $f(x+y) - f(x-y) = 4xy$.

3. Given that $f(ax) = a\,f(x)$ for all real a and x, and that $f(2) = 5$, find $f(17)$.

4. Given $f(x)$ such that $f(1-x) + (1-x)\,f(x) = 5$, find $f(5)$.

5. (a) Is the largest possible domain of $fg(x)$ the same as the largest possible domain of $g(x)$?

(b) What are necessary and sufficient conditions for the function $fg(x)$ to exist?

6. Prove that either $g(x) = f(x)$ or $g(x) = f^{-1}(x)$ is a sufficient but not necessary condition for $fg(x) = gf(x)$.

7. (a) What values of a and b make $f(x) = \dfrac{x-a}{x-b}$ a one to one function?

(b) What values of a and b make $f(x) = \dfrac{x-a}{x-b}$ a self inverse function?

8. Prove that $(f \circ g)^{-1}(x) = g^{-1} \circ f^{-1}(x)$.

Extension worksheet 6 Transformations of graphs: modulus functions

Questions

1. (a) Sketch the graph of $|x|+|x-3|$.

   (b) Solve the equation $|x|+|x-3|=5$.

2. Solve the equation $|x|+|x+1|+|x+3|=7$.

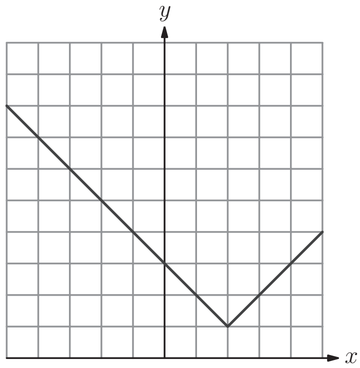
3. Solve the equation $|2+|x||=5$.

4. Describe the graph

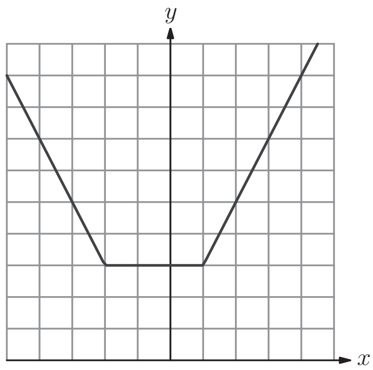
   $$y=\begin{cases}3-x & x<3\\ x-3 & x\geq 3\end{cases}$$

   as a function with the same rule across all of its domain.

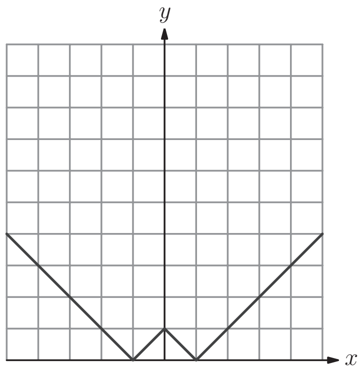
5. Describe the graph below as a function with the same rule across all of its domain.



6. Describe the following graph as a function with the same rule across all of its domain.



7. Find the equation of the following graph as a function with the same rule across all of its domain.



Extension worksheet 7 Sequences and series

Questions

1. A textbook writer wants to set a question of the type:

The sum to infinity of a geometric series is 1 and the second term of the geometric series is x. Find the value of the common ratio.

He wants to select a value of x so that there is only one answer to this question. What are the possible values of x?

2. The sum of the first p terms of an arithmetic series is q and the sum of the first q terms of the same arithmetic series is p, where $p \neq q$. Find the sum of the first $(p + q)$ terms of the series. (*© Ed excel 2010*)

3. (a) Express the decimal $a = 0.7777\ldots$ as a fraction.

(b) If $u_1 = 7$, $u_2 = 77$, $u_3 = 777$ $u_n = \underbrace{77\ldots77}_{n \text{ digits}}$ express u_n in terms of a.

(c) Hence or otherwise find an expression for the sum

$S_n = 7 + 77 + 777 + 7777\ldots(n \text{ terms})$.

4. Prove that $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\ldots$ does not converge.

> This sequence is called the harmonic sequence and it has some very interesting properties.

Extension worksheet 8 Binomial expansion

Questions

Prove the following binomial identities. Try:

- considering the factorial form of binomial coefficients
- considering binomial expansions with clever values substituted in
- comparing coefficients of expressions involving binomial coefficients.

Several require previous results.

1. $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$

2. $\binom{n}{r} = \binom{n}{n-r}$

3. $\binom{n}{r} r = n\binom{n-1}{r-1}$

4. $\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-r}{m-r}$

5. $\sum_{r=0}^{r=n}\binom{n}{r} = 2^n$

6. $\sum_{r=0}^{r=n}(-1)^r\binom{n}{r} = 0$

7. $\sum_{r=0}^{r=n}\binom{k+r}{r} = \binom{k+n+1}{n}$

8. $\sum_{r=0}^{r=n}\binom{n}{r}\binom{m}{k-r} = \binom{n+m}{k}$

9. $\sum_{r=0}^{r=n}\binom{n}{r}^2 = \binom{2n}{n}$

For questions $1-8$ try proving the identities using combinatorial arguments.

Extension worksheet 9 Circular measure and trigonometric functions

Throughout this worksheet, you should use the following definition of a periodic function:

The function $f(x)$ is periodic with period p if, for all x, $f(x+p)=f(x)$.

Questions

1. (a) Use your calculator to sketch the graph of $y=\sin x+\sin 3x$ (x is in radians).

(b) Use your graph to estimate the period of $f(x)=\sin x+\sin 3x$.

(c) Use the above definition to prove that $f(x)$ is a periodic function.

2. Use your calculator to estimate the periods of the following functions, then prove your estimate:

(a) $\sin x+\sin 5x$

(b) $\sin 2x+\sin 4x$

(c) $\sin 2x+\sin 3x$

(d) $\sin 4x+\sin 6x$

3. State and prove a general result for the period of $f(x)=\sin mx+\sin nx$ where $m,n\in\mathbb{N}$.

4. Use your calculator to estimate the periods of the following functions, then prove your estimate:

(a) $\sin x+\sin\left(\frac{x}{3}\right)$

(b) $\sin\left(\frac{x}{2}\right)+\sin\left(\frac{x}{4}\right)$

(c) $\sin\left(\frac{x}{2}\right)+\sin\left(\frac{x}{3}\right)$

(d) $\sin\left(\frac{x}{4}\right)+\sin\left(\frac{x}{6}\right)$

5. State and prove a general result for the period of $f(x)=\sin\left(\frac{x}{m}\right)+\sin\left(\frac{x}{n}\right)$ where $m,n\in\mathbb{N}$.

6. (a) State the period of $\sin(\pi x)$.

(b) Use your calculator to sketch the graph of $y=\sin x+\sin(\pi x)$. Is this a periodic function?

Extension worksheet 10 Trigonometric equations and identities

You should attempt all the questions without a calculator, but you can use the calculator to check your answers.

A trigonometric equation like $\sin x = \frac{1}{2}$ has infinitely many solutions.

Two of the solutions are : $x_1 = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ and $x_2 = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

All other solutions can be found by adding and subtracting multiples of 2π. We can write this as

$$x_1 = \frac{\pi}{6} + 2k\pi,\ x_2 = \frac{5\pi}{6} + 2k\pi \qquad (k \in \mathbb{Z})$$

This is called the *general solution* of the equation. By selecting particular values of the integer k we obtain particular solutions in a given interval.

Questions

1. Find the general solution of the following equations, and hence find all the solutions in the interval $[0, \pi]$:

(a) $\tan 3x = \sqrt{3}$

(b) $\sin x = \sin(1.2)$

(c) $\sin x = \cos(1.2)$

(d) $\cos x = -\sin(0.3)$

(e) $\sin x = \sin 2x$

(f) $\cos x = \cos\left(2x - \frac{\pi}{4}\right)$

2. (a) Given that the equation $\sin x = 0.3$ has two solutions in the interval $[0, k]$, how many solutions can it have in the interval $[0, 2k]$?

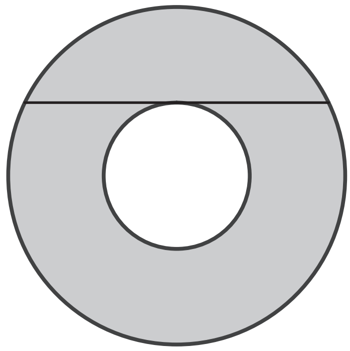
(b) Given that the equation $\sin x = 0.3$ has two solutions in the interval $[0, k]$, how many solutions can the equation $\sin(2x) = 0.3$ have in the interval $[0, k]$?

3. Find the value of $\tan 10° \tan 20° \tan 30° \ldots \tan 80°$.

Extension worksheet 11a Geometry of triangles and circles

Questions

1. The diagram shows two concentric circles. The chord of the large circle is tangent to the small circle and has length $2p$. What is the area of the shaded region between the two circles?



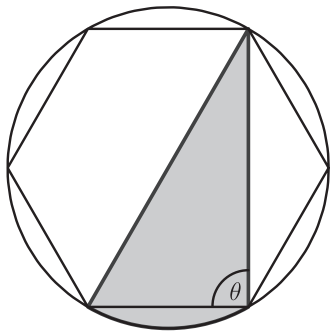
2. (a) Find the length of the side of a regular hexagon inscribed in a circle of radius r.

   (b) Show that the area of an equilateral triangle of side a is $\dfrac{a^2\sqrt{3}}{4}$.

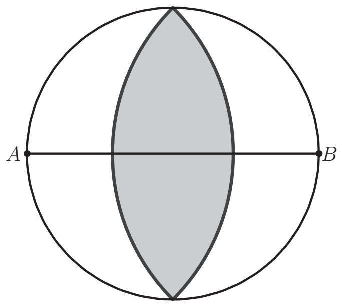
   (c) The diagram shows a regular hexagon inscribed in a circle of radius r.

   (i) Write down the size of angle θ in radians.

   (ii) Find the exact area of the shaded region.



3.



AB is a diameter of a circle of radius r. Two circular arcs of equal radius are drawn with centres A and B. The arcs meet on the circle, as shown. Find the shaded area.

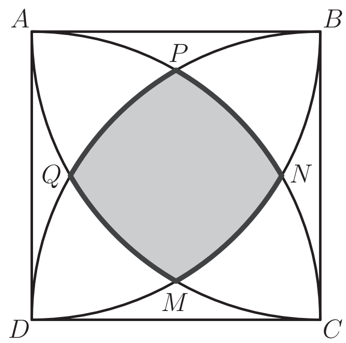
4. The diagram shows a square $ABCD$ of side a and four circular arcs centred on each of the vertices.

   (a) Explain why triangle ABM is equilateral.

   (b) Find the size of the angle $M\hat{A}N$.

   (c) Find the perimeter of the shaded region $MNPQ$.

   (d) Find the area of the shaded region.



Extension worksheet 11b Geometry of triangles and circles

Questions

1. Consider an isosceles triangle ABC with base $BC = 1$, $AC = BC$ and $\hat{A} = 36°$. Draw a bisector of angle $\hat{B}$ and let it intersect AC at D.

(a) By using similar triangles, or otherwise, find CD.

(b) Use the cosine rule to find the exact value of $\cos 36°$.

2. Triangle ABC is inscribed in a circle of radius R (this is called the circumcircle of the triangle).

(a) By drawing a line from B through the centre of the circle, which intersects the circle at D, use triangle BCD to express BC in terms of R and $\hat{A}$.

(b) Deduce the Extended Sine Rule: $\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}} = 2R$.

(c) Hence show that the area of the triangle ABC is given by $2R^2 \sin \hat{A} \sin \hat{B} \sin \hat{C}$.

3. A walking party observe two lakes, R and T, in the distance. They want to measure the straight line distance between the two lakes. To do this, they set up two observation points, A and B, 500 m apart. The four points all lie in the same plane.

The angles from A and B to the lakes are then measured:

$A\hat{B}R = 21°$, $A\hat{B}T = 145°$, $B\hat{A}R = 155°$, $B\hat{A}T = 33°$.

Find, to the nearest 10 m, the distance between the two lakes.

4. Suppose that in the previous question, R and T are mountain peaks, so that A, B, R and T are not necessarily in the same plane. Explain why the above information is not sufficient to find the distance between R and T. (Where does your method fail?)

What additional measurements are needed?

5. One astronomical unit is defined to be the distance of the Earth from the Sun ($1\text{au} \approx 149.6 \times 10^6$ km). Assume that the length of one Earth year is 365 days.

A simple astronomical model has Earth and Mars rotating around the Sun, S, in circular orbits (Mars in the outer orbit), with the radius of the Earth's orbit being 1au. At a point A in Earth's orbit, an astronomer observes Mars, M, to be located so that $S\hat{A}M = 116.5°$. It is known that Mars takes 687 (Earth) days to orbit the Sun. After 687 days, when the Earth is in position B, the astronomer observes Mars again, and finds that $S\hat{B}M = 140°$.

Use this data to find the distance from the Sun to Mars, in au.

(Note: It may not be immediately obvious how to draw the diagram, but when you start calculating angles, you will find that there is only one arrangement of the four points that works.)

Extension worksheet 12a Further trigonometry

Questions

1. Find the exact value of $\cos 20° \cos 40° \cos 80°$.
2. Show that $\tan 50° \tan 60° \tan 70° = \tan 80°$.
3. Show that $\cos x = \dfrac{1-t^2}{1+t^2}$ and $\sin x = \dfrac{2t}{1+t^2}$ where $t = \tan\dfrac{x}{2}$.
4. (a) Show that $\sin P + \sin Q = 2\sin\left(\dfrac{P+Q}{2}\right)\cos\left(\dfrac{P-Q}{2}\right)$.

   (b) Find the exact value of $\sin 75° + \sin 15°$.

   (c) If $\hat{A}$, $\hat{B}$ and $\hat{C}$ are angles in a triangle, show that $\sin\hat{A} + \sin\hat{B} + \sin\hat{C} = 4\cos\dfrac{\hat{A}}{2}\cos\dfrac{\hat{B}}{2}\cos\dfrac{\hat{C}}{2}$.
5. Solve the equation $\cos x + \cos 2x + \cos 3x = 0$ for $x \in [0, 2\pi]$.
6. If x, y and z are angles in a triangle, show that

   $\cot x \cot y + \cot y \cot z + \cot z \cot x = 1$.
7. (a) Show that $\left(\sqrt{2}+1\right)^2 = 3 + 2\sqrt{2}$.

   (b) Express $\cos 3\theta$ in the form $\cos\theta(a - b\sin^2\theta)$.

   (c) Solve the equation $\sqrt{2}\left(\sin 2\theta + \cos\theta\right) + \cos 3\theta = \sin 2\theta + \cos\theta$
   for $\theta \in [0°, 360°]$.

Extension worksheet 12b Further trigonometry

Questions

1. Solve the following equation, for $0 \le x \le \pi$, giving your answers in terms of π:

$\sin 5x - \cos 5x = \cos x - \sin x.$

(© *Ed excel 2002*)

2. Find the values of $\tan\theta$ such that $2\sin^2\theta - \sin\theta\sec\theta = 2\sin 2\theta - 2$.

(© *Ed excel 2003*)

3. Solve for $0 < \theta < 2\pi$

$\sin 2\theta + \cos 2\theta + 1 = \sqrt{6}\cos\theta,$

giving your answers in terms of π. (© *Ed excel 2005*)

4. Solve the equation $\cos x + \sqrt{\left(1 - \frac{1}{2}\sin 2x\right)} = 0$ in the interval $0° \le x < 360°$.

(© *Ed excel 2004*)

5. Given that $(\sin\theta + \cos\theta) \ne 0$, find all the solutions of

$$\frac{2\cos 2\theta(\sin 2\theta - \sqrt{3}\cos 2\theta)}{\sin\theta + \cos\theta} = \sqrt{6}\,(\sin 2\theta - \sqrt{3}\cos 2\theta) \text{ for } 0 \le \theta < 360°.$$ (© *Ed excel 2006*)

6. (a) Solve, for $0 \le x < 2\pi$,

$\cos x + \cos 2x = 0.$

(b) Find the exact value of x, $x \ge 0$, for which

$\arccos x + \arccos 2x = \frac{\pi}{2}.$ (© *Ed excel 2007*)

7. (a) Prove that $\tan 15° = 2 - \sqrt{3}$.

(b) Solve, for $0 \le \theta < 360°$, $\sin(\theta + 60°)\sin(\theta - 60°) = (1 - \sqrt{3})\cos^2\theta$.

(© *Ed excel 2008*)

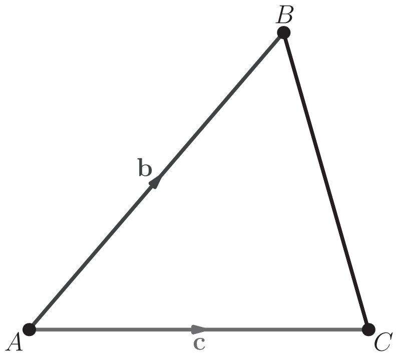
8. Solve the equation $\arcsin(1-2x) = \frac{\pi}{3} - \arcsin x$. (© *Ed excel 2009*)

Extension worksheet 13 Vectors

This worksheet will show you some examples of using vectors to prove geometrical facts. You may be familiar with the facts already, and may have proved them using other methods, but vectors provide short and elegant proofs.

Questions

In all three questions, ABC is a triangle and we define vectors $\boldsymbol{b} = \overrightarrow{AB}$ and $\boldsymbol{c} = \overrightarrow{AC}$. It follows that $\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b}$.

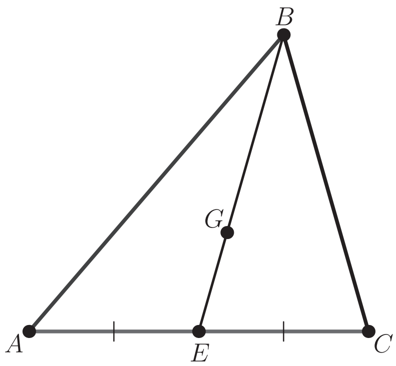


1. We will prove that the line connecting the midpoints of two sides of a triangle is parallel to and half the length of the third side.

Express $\overrightarrow{MN}$ in terms of $\boldsymbol{b}$ and $\boldsymbol{c}$, where M is the midpoint of AB and N is the midpoint of AC.

How does this imply that $MN \parallel BC$ and $MN = \frac{1}{2}BC$?

2. We will prove that the medians of a triangle all pass through the same point, which divides each median in the ratio $2:1$.



(a) Let E be the midpoint of AC. The line BE is called a median.

(i) Express $\overrightarrow{BE}$ in terms of $\boldsymbol{b}$ and $\boldsymbol{c}$.

(ii) Let G be the point on BE such that $BG : GE = 2:1$. Express $\overrightarrow{BG}$ in terms of $\boldsymbol{b}$ and $\boldsymbol{c}$.

(b) (i) Express $\overrightarrow{CG}$ in terms of $\boldsymbol{b}$ and $\boldsymbol{c}$.

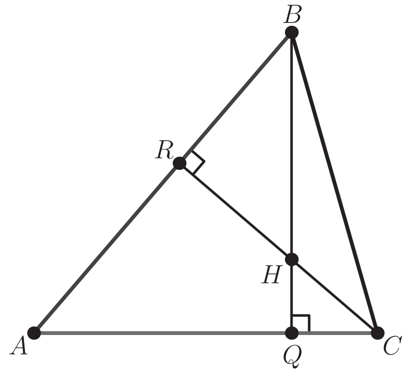
(ii) Let F be the midpoint of AB. Express $\overrightarrow{CF}$ in terms of $\boldsymbol{b}$ and $\boldsymbol{c}$.

(iii) Hence explain why G lies on CF, and $CG : GF = 2:1$.

(c) Conclude that all three medians pass through the point G, which is called the *centroid* of the triangle.

3. We will prove that all three altitudes of a triangle pass through the same point.

Let Q and R be the points on sides AC and AB, respectively, such that BQ is perpendicular to AC and CR is perpendicular to AB. These lines are called *altitudes*. Let H be the intersection of BQ and CR.



(a) Explain why $\overrightarrow{BH}\cdot\boldsymbol{c}=0$ and $\overrightarrow{CH}\cdot\boldsymbol{b}=0$.

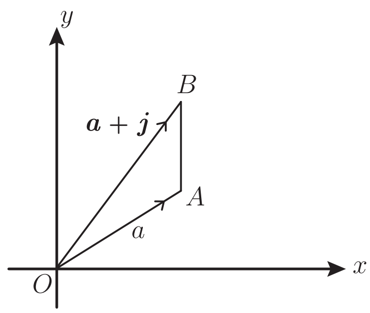
(b) You can see from the diagram that $\overrightarrow{AH}=\boldsymbol{b}+\overrightarrow{BH}$. Write a second expression for $\overrightarrow{AH}$ involving vector $\boldsymbol{c}$.

(c) Complete the following proof that $\overrightarrow{AH}\cdot(\boldsymbol{c}-\boldsymbol{b})=0$:

$\overrightarrow{AH}\cdot(\boldsymbol{c}-\boldsymbol{b})=\overrightarrow{AH}\cdot\boldsymbol{c}-\overrightarrow{AH}\cdot\boldsymbol{b}$	properties of scalar product
$=\left(\boldsymbol{b}+\overrightarrow{BH}\right)\cdot\boldsymbol{c}-(\qquad\qquad)\cdot\boldsymbol{b}$	from (b)
$=\boldsymbol{b}\cdot\boldsymbol{c}+\overrightarrow{BH}\cdot\boldsymbol{c}-$ _____	properties of scalar product
$=\boldsymbol{b}\cdot\boldsymbol{c}-\boldsymbol{c}\cdot\boldsymbol{b}$	from _____
$=0$	

(d) Conclude that AH is also an altitude, and hence all three altitudes pass through point H, which is called the *orthocentre* of the triangle.

4.



The point A is a distance 1 unit from the fixed origin O. Its position vector is $\boldsymbol{a}=\frac{1}{\sqrt{2}}(\boldsymbol{i}+\boldsymbol{j})$.

The point B has position vector $\boldsymbol{a}+\boldsymbol{j}$.

By considering ΔOAB, prove that $\tan\frac{3\pi}{8}=1+\sqrt{2}$.

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Extension worksheet 14 Lines and planes in space

Questions

1. Ship S_1 sails from a port located at the origin and moves with velocity $\begin{pmatrix}3\\1\end{pmatrix}$.

   (a) Write down the position vector of S_1 at time t.

   At the same time, ship S_2 starts from the point $(5,-2)$ and moves with velocity $\begin{pmatrix}k\\4\end{pmatrix}$.

   (b) Write down the position vector of S_2 at time s.

   (c) Find the range of values of k for which the paths of the two ships cross (but they do not necessarily meet). Remember that $t,s \geq 0$. (Hint: you can solve this problem graphically as well as algebraically.)

   (d) Find the value of k for which the two ships will collide.

2. Two birds fly in a straight line.

   Their paths have equations $\boldsymbol{r}_1 = \begin{pmatrix}0\\0\\0\end{pmatrix} + t\begin{pmatrix}3\\1\\5\end{pmatrix}$ and $\boldsymbol{r}_2 = \begin{pmatrix}3\\-1\\4\end{pmatrix} + s\begin{pmatrix}a\\b\\2\end{pmatrix}$, with $s,t \geq 0$.

   (a) Given that their paths cross, show that $3a+b=4$.

   (b) Find another condition satisfied by a and b, given that $a,b \geq 0$. (Hint: use $s,t \geq 0$.)

   (c) Find the values of a and b for which the two birds meet.

3. (a) Lines l_1 and l_2 have equations $\boldsymbol{r} = \begin{pmatrix}0\\-4\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix}1\\3\\5\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\p\end{pmatrix}$. Find the value of p for which the lines intersect.

   Use the above value of p for the rest of this question.

   (b) Line l_3 has equation $\boldsymbol{r} = \begin{pmatrix}0\\-4\\1\end{pmatrix} + t\begin{pmatrix}a\\3\\2\end{pmatrix}$.

   (i) Show that l_3 and l_1 intersect for all values of a.

   (ii) Find the value of a for which l_3 also intersects l_2.

   In the next part, it is given that l_3 intersects both l_1 and l_2.

   (c) Line l_4 intersects both l_1 and l_2. None of the four lines are parallel to each other. Show that l_4 also intersects l_3.

4. Four snails move in the same plane, each one travelling in a straight line. None of the paths are parallel to each other.

   Given that five of the six possible meetings have already occurred, prove that the sixth meeting must also occur.

Extension worksheet 15 Complex numbers

Loci in the complex plane

A locus is a description of points which obey a rule.

In Cartesian form, complex numbers act just like vectors. In particular, the construction $z_1 - z_2$ describes a vector from z_2 to z_1. So the equation $|z-(2+\mathrm{i})|=3$ can be described in words as 'the distance of z from the point $2+\mathrm{i}$ is 3'. On the Argand diagram this describes a circle, centred on $2+\mathrm{i}$ with radius 3.

Describe the shapes formed by the following rules. In some of the questions, just considering what the equation represents will be enough. In others you may need to consider the real and imaginary parts of $z = x + \mathrm{i}y$ and form an equation in x and y.

Questions

1. $|z-1|=|z-\mathrm{i}|$

2. $|z-1|=2|z|$

3. $|z-1|=k|z|$

4. $\arg(z-\mathrm{i})=\dfrac{\pi}{3}$

5. $\arg(z-\mathrm{i})=\arg(z+1)$

6. $\arg(z-\mathrm{i})=\arg(z+1)+\pi$

7. $\arg(z-\mathrm{i})=\arg(z+1)+\dfrac{\pi}{2}$

8. $\arg z=|z|$

9. $\arg z=\arg z^*$

Extension worksheet 16 Basic differentiation and its applications

Questions

1. The points P and Q lie on the graph of $y = x^2$ such that the x-coordinate of P is a and the x-coordinate of Q is $-a$ $(a > 0)$. The normals at the points P and Q meet at M. Show that the vertical height of triangle PQM is independent of a.

2. The point M on the graph of $y = x^2$ is vertically below the point N on the graph of $y = \sqrt{x}$. If the tangents at M and N never meet, find the value of the x-coordinate of M.

3. S and T are two points on the parabola $y = x^2$. The tangents at these two points meet at the point (4,15). Find the coordinates of S and T.
   If the tangents to S and T meet at (j, k) prove that $j^2 > k$.

4. Point P lies on the graph of $y = \sqrt{x}$ and has x-coordinate a. The tangent to the curve at P crosses the x-axis at M and the y-axis at Q. The normal to the curve at P crosses the x-axis at N and the y-axis at R. Find the ratio of the areas of triangles PQR and PMN in terms of a.

5. Point $A\left(a, \frac{1}{a}\right)$ lies on the hyperbola $y = \frac{1}{x}$. The normal to the hyperbola at A intersects the hyperbola again at B.
   (a) Find the coordinates of B.
   (b) Find the equation of the tangent to the hyperbola at B.

6. $f(x) = ax^3 + x^2 + bx + c$, where a, b and c are positive constants.
   The graph of $y = f(x)$ intersects the y-axis at the point P. The tangent at P to the curve $y = f(x)$ meets the curve again at the point Q.
   (a) Find, in terms a, b and c, the coordinates of Q.
   Given that the normal at P to the curve $y = f(x)$ is also a tangent at the point R to the curve:
   (b) determine an expression for a in terms of b
   (c) obtain, in terms of b and c, the coordinates of R.
   Given also that $PQ = 4PR$:
   (d) find the values of a and b. (© *Ed excel 1993*)

Extension worksheet 17 Basic integration

Questions

1. Evaluate the following integrals:

(a) $\int_{-3}^{3} \left||x|-1\right| \mathrm{d}x$

(b) $\int_{-3}^{3} \left|\left||x|-1\right|-1\right| \mathrm{d}x$

2. The floor function $\lfloor x \rfloor$ (also called the integer part of x) is defined to be the largest integer which is not larger than x. So for example, $\lfloor 3.2 \rfloor = 3$, $\lfloor 5 \rfloor = 5$ and $\lfloor -4.2 \rfloor = -5$.

(a) Show that $\int_0^4 \lfloor x \rfloor \mathrm{d}x = 6$.

(b) Find an expression for $\int_0^M \lfloor x \rfloor \mathrm{d}x$ for $M \in \mathbb{N}$.

(c) Find $\int_0^a \lfloor x \rfloor \mathrm{d}x$ where $a \in \mathbb{R}, a > 0$.

3. The function $h(x)$ is defined for $x \geq 0$ by

$$h(x) = \frac{1}{2^n} \text{ for } x \in [n-1, n[$$

(a) Show that $\int_0^M h(x)\,\mathrm{d}x = \frac{2^M - 1}{2^M}$ for $\mathrm{M} \in \mathbb{N}$.

(b) Evaluate $\int_0^\infty h(x)\,\mathrm{d}x$.

4. (a) Find the area between the curve $y = \sin x$ and the x-axis, bounded by $x = 0$ and $x = 2\pi$.

(b) By considering the transformation which maps the graph of $y = \sin x$ to the graph of $y = \sin 2x$, find the area between the curve $y = \sin 2x$ and the x-axis bounded by $x = 0$ and $x = 2\pi$.

(c) Evaluate $\int_0^{\pi/n} \sin(nx)\,\mathrm{d}x$ where $n > 0$.

5. The average value of the function $f(x)$ over the interval $[a, b]$ is defined to be $\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$.

(a) Find the average values of the following functions over the given interval:

(i) x^2 over $[0, 4]$

(ii) e^x over $[0, 1]$

(iii) $\sin x$ over $[0, 2\pi]$

(iv) $|\sin x|$ over $[0, 2\pi]$

(b) Show that if $f(x)$ has the same average over any interval $[0, a]$ then f is a constant function.

Extension worksheet 18 Further differentiation methods

Questions

1. Leibniz's rule is an extension of the product rule to higher order derivatives:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(f(x)g(x)\right)=\sum_{k=0}^{v=n}\binom{n}{k}\frac{\mathrm{d}^k f}{\mathrm{d}x^k}\frac{\mathrm{d}^{n-k} g}{\mathrm{d}x^{n-k}}$$

(a) Find the fourth derivative of $\mathrm{e}^x \sin x$ with respect to x.
If $Z(x) = f(x)\,g(x)$:

(b) If $f(x)$ and $g(x)$ both have a minimum point at $x = a$, prove that $z(x)$ has a minimum point if $g(a) > -\dfrac{g''(a)f(a)}{f''(a)}$.

(c) If $f(x)$ and $g(x)$ both have a maximum point at $x = a$, show that the same condition implies that $z(x)$ has a maximum point.

(d) If $f(x)$ and $g(x)$ both have a stationary point of inflexion at $x = a$, show that $z''(a) = 0$. By providing a counterexample, prove that in these conditions $z(x)$ does not necessarily have a point of inflexion at $x = a$.

(e) If $z(x)$ has a stationary point of inflexion at $x = a$, show that $f(a) = \dfrac{f''(a)g(a)^2}{2g'(a)^2 - g(a)g''(a)}$.

2. (a) Find the maximum value of $y = x^{1/x}$ for $x > 0$.

(b) Sketch the graph of $y = x^{\frac{1}{x}}$ for $x > 0$.

(c) Find an expression for $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$.

(d) Show that the graph $y = x^{1/x}$ has a point of inflexion between $x = \dfrac{1}{\mathrm{e}}$ and $x = 1$.
(You do not need to find the coordinates of this point.)

Extension worksheet 19 Further integration methods

Questions

1. Using the substitution $u = \frac{\pi}{2} - x$, prove that $\int_0^{\pi/2} \frac{1}{1+\tan^k x}\,dx = \frac{\pi}{4}$.

2. Using the substitution $u = \frac{1}{x}$, prove that the area under the curve $y = \frac{x+1}{x^3+1}$ between 0 and 1 is half of the area under the curve between 0 and ∞.

3. $I = \int \frac{1}{(1-x)\sqrt{(x^2-1)}}\,dx,\ x > 1.$

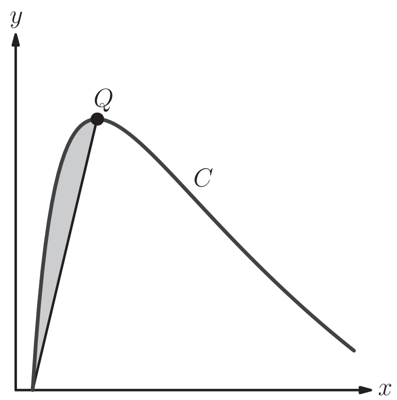
(a) Use the substitution $x = 1 + u^{-1}$ to show that $I = \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}} + c$.

(b) Hence show that

$$\int_{\sec\alpha}^{\sec\beta} \frac{1}{(1-x)\sqrt{(x^2-1)}}\,dx = \cot\left(\frac{\alpha}{2}\right) - \cot\left(\frac{\beta}{2}\right),\ 0 < \alpha < \beta < \frac{\pi}{2}.$$

(*© Ed excel 2010*)

4.



The graph shows a sketch of part of the curve C with equation $y = \sin(\ln x)$, $x \geq 1$.

The point Q, on C, is a maximum.

(a) Show that the point $P(1, 0)$ lies on C.

(b) Find the coordinates of the point Q.

(c) Find the area of the shaded region between C and the line PQ.

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Extension worksheet 20a Further applications of calculus

Questions

1. The centre of mass of an object is the point at which all the weight appears to act (for the purposes of balancing). Centres of mass of some objects can be found using integration.

For a volume of revolution formed by rotating the part of the curve with equation $y = f(x)$ between $x = a$ and $x = b$ about the x-axis, the centre of mass has coordinates $(\bar{x}, 0)$, where

$$\bar{x} = \frac{\int_a^b xy^2 \, dx}{\int_a^b y^2 \, dx}$$

(a) A hemisphere of radius r is formed by rotating a part of the circle $x^2 + y^2 = r^2$ about the x-axis.

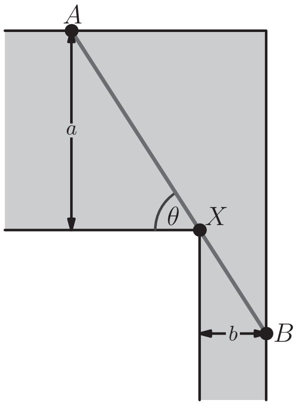
Show that the distance of the centre of mass from the base of the hemisphere is $\frac{3}{8}r$.

(b) A cone of radius r and height h is formed by rotating a part of the line $y = mx$ about the x-axis.

(i) Express m in terms of r and h.

(ii) Find the distance of the centre of mass from the base of the cone.

2. A ladder AB is carried around a corner from a corridor of width a into a corridor of width b, as shown in the diagram.



AXB is a straight line making angle θ with the first corridor, as shown.

(a) Write down an expression for the length AB in terms of a, b and θ.

(b) Show that the maximum length of the ladder that can be carried around the corner is

$$\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

3. Given that $\frac{d}{dx}(u\sqrt{x}) = \frac{du}{dx} \times \frac{d(\sqrt{x})}{dx}$, $0 < x < \frac{1}{2}$,
where u is a function of x, and that $u = 4$ when $x = \frac{3}{8}$, find u in terms of x.

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Extension worksheet 20b Further applications of calculus

Introduction to differential equations

In real life there are many situations where it is easier to measure how fast something is changing rather than its actual value. For example, finding the volume of water in a river is hard, but finding the rate of water flowing past a point is much easier. Whenever we are working with rates, we interpret this mathematically as a derivative. If we have an equation involving a rate, we call this a differential equation. For example:

$$\frac{dy}{dx} = 2y.$$

When we are asked to solve a differential equation it means finding all possible functions which work. In the example above, the solution would be any function of the form $y = ke^{2x}$ where k is a constant.

One particularly important differential equations looks like

$$\frac{d^2y}{dx^2} = -\omega^2 y.$$

Where ω is a constant. You can check that a solution to this equation is $A\sin(\omega x)$. This differential equation leads to oscillating behaviour called simple harmonic motion.

Questions

1. Use the fact that integration is the opposite of differentiation to solve the following differential equations:

(a) $\frac{dy}{dx} = 1 + x$

(b) $\frac{dy}{dx} = \sin x$

(c) $\frac{dy}{dx} + x - x^2 = 0$

2. By inspection, find a solution to the following differential equations:

(a) $\frac{dy}{dx} + y = 0$

(b) $\frac{d^2y}{dx^2} + 4y = 0$

(c) $\frac{dy}{dx} = \frac{1}{2y}$

3. By trying a function of the form $y = e^{kx}$, suggest solutions to the differential equation:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 6y = 0$$

4. A spring exerts a force $F = -kx$ when it is extended a distance x beyond its equilibrium point. Use Newton's second law $\left(F = m\frac{d^2x}{dt^2}\right)$ to find a differential equation satisfied by x.

Hence describe the motion of the spring and find the frequency of the motion.

Extension worksheet 21 Summarising data

Questions

1. An entire population is made up of the numbers $1, 2$ and 3.

(a) Find the mean and standard deviation of this population.

(b) List all possible samples of size two that can be drawn from this population if repetition is not allowed.

(c) Find the mean and variance of each of these samples.

(d) Find the mean of the sample means and compare to the population mean.

(e) Find the mean of the sample variances and compare to the population variance.

(f) Repeat parts (b) to (e) for all possible samples of size two if repetition is allowed.

In the following two parts we consider only samples where repetition is not allowed (i.e. all the elements of the sample must be different).

(g) Prove that for a population of size $n+1$ the mean sample mean of a sample of size n is μ.

(h) Prove that for a population of size $n+1$ the mean sample variance of a sample of size n is $\dfrac{n^2-1}{n^2}\sigma^2$.

Extension worksheet 22 Probability

Questions

1. A point P is chosen at random inside the square $ABCD$. Find the probability that the angle APB is obtuse.
2. Two trains are both equally likely to arrive at a station at any time between 11:00 and 11:20. Their arrival time is independent. Find the probability that they arrive within five minutes of each other.
3. A mother has two children who are not twins. One of them is a boy born on a Tuesday. What is the probability that she has two boys?

> The ability to be able to distinguish between two states is important in both probability and physics. Have a look at Pauli's Exclusion Principle.

4. Consider a sequence of three letters which are randomly either A or B (e.g. ABA).
   (a) Show that the probability of the sequence 'AA' occurring is $\frac{3}{8}$.
   (b) Find the probability of the sequence 'AB' occurring.
   (c) In a sequence of 10 As and Bs, which sequence is more likely to be found first: ABA or AAB?

> The result of the above question is important in designing the technology for genetic sequencing. You may want to have a look at the mathematics involved in polymerase chain reaction.

Extension worksheet 23 Discrete probability distributions

Questions

1. (a) Show that there is no probability distribution with possible outcomes 1,2, and 3 which has mean and variance equal to 1.

(b) The variable X can take the distinct values 0,1 and y. Find the smallest possible value of y such that $\mathrm{E}(X)=1$ and $\mathrm{Var}(X)=1$.

2. Initially there is a 50% chance of Gary winning a point in tennis. If Gary wins a point there is a 75% probability that he will win the next point. If he loses a point there is a 60% chance that he will lose the next point.

(a) Find the probability that Gary wins the third point.

(b) What is the expectation and variance of the number of points that he wins in the first four points?

> This is an example of a *Markov Chain*, a branch of probability dealing with a series of dependent events.

3. The numbers of girls in a club (G) and the number of boys in a club (B) have the following probability distribution:

		G			
		1	2	3	4
B	1	0.01	0.05	0.06	0.01
	2	0.03	0.13	0.10	0.06
	3	0.05	0.10	0.16	0.06
	4	0.01	0.06	0.08	0.03

(a) Find $\mathrm{P}(G=1)$

(b) Find $\mathrm{P}(B=2)$

(c) Find $\mathrm{P}(G=1 \mid B=2)$

(d) Find $\mathrm{E}(G)$ and $\mathrm{Var}(G)$

(e) Find $\mathrm{E}(B)$ and $\mathrm{Var}(B)$

(f) If $T=$ ‘total number of people in the club’, find E(T) and Var(T)

> This is an example of a *bivariate distribution*.

Extension worksheet 24 Continuous distributions

Questions

1. A set of vertical parallel lines is drawn, with 2 cm separation between lines. A needle of length 2 cm is dropped onto these lines. The position of the centre of the needle and the angle it makes are both random with all values being equally likely.
   (a) If the distance of the centre of the needle to the nearest line is given by x cm, show that the probability that the needle touches the line is given by $\frac{2\arccos x}{\pi}$.
   (b) Given that the probability of being within x cm of the nearest line is x, independent of the angle that the needle is lying at, find the probability that the needle crosses the line.

> This is a famous problem called the 'Buffon's needle'. This method gives us an experimental way of determining π, using something called Monte Carlo simulations.

2. Given that the Z distribution has probability density function $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}$, prove that the mean of the Z distribution is 0 and the variance is 1.

3. The Cauchy distribution has a probability density function $\frac{1}{k(1+x^2)}$ for all x.
   (a) Find the value of the constant k.
   (b) Sketch the Cauchy distribution.
   (c) Find the mean of the Cauchy distribution.
   (d) Find the variance of the Cauchy distribution.

> If you study Option 7, Statistics, you will meet a result called the Central Limit Theorem, which states that if many observations of a random variable are taken, their mean follows a normal distribution. However, this result only applies to random variables whose distribution has a finite variance. The Cauchy distribution is an example where the Central Limit Theorem does not apply. In fact, if many observations of a Cauchy random variable are taken, their mean follows another Cauchy distribution.

4. An exponential distribution has the probability density function $f(x) = 3\mathrm{e}^{-3x}$ for $x > 0$.
   (a) Find the mean of this distribution.
   (b) Find the variance of this distribution.
   (c) If Y follows this distribution, find $\mathrm{P}(Y < 5)$.
   (d) Find $\mathrm{P}(Y < a+5 \mid Y > a)$ and comment on your result.

Answers: Extension worksheets

1 Counting principles

1. 8.

The fact that patterns lie in a loop poses a challenge. We can eliminate the problem by breaking into cases: either the pattern can have two whites together or it can't.

If two whites can NEVER be adjacent, the pattern is fixed as alternating BWBWBWBW (only one possible pattern).

If two whites can be adjacent, we can pull them to the 'front' of the line, and then consider the problem as being one of inserting 0,1 or 2 whites into 4 blacks.

0 Whites in: pattern is WWWWBBBB (or equivalent): one pattern

1 White in: pattern is WWWBWBBB or WWWBBWBB: two patterns (note that WWWBBBWB is the same as the first one listed here, since the loop need only be turned over).

2 Whites in: pattern is WWBWWBBB / WWBWBWBB / WWBBWWBB / WWBWBBWB

2. 2601.

One way to approach this problem is to restate it as 'In how many ways can you pick two positive integers which sum to an even number less than 100?'.

The first integer is the number of £1 coins, and the second is half the number of 50p coins (there must be an even number of these, so halving it is valid). £2 coins will make up the remainder automatically.

0 can be split 1 way

2 can be split 3 ways

4 can be split 5 ways, etc.

We quickly see that the solution is $1 + 3 + 5 + \cdots + 101 = 2601$

3. 1092, every player apart from the eventual winner is eliminated in one match.

4. (a) 15
(b) 264
(c) 265

Some research on derangements may be needed to explain these answers.

5. (a) $\binom{9}{6} = 84$

Each of the six marks must be different and chosen from 0 to 8; the ordering is automatic.

(b) 3003

There could be 1,2,3,4,5 or 6 different marks, each determined in $\binom{9}{r}$ ways.

There is one way to make 5 ties, 5 ways to arrange four ties to 2 marks, 10 ways to arrange three ties to 3 marks, 10, 5 and 1 way to arrange no ties for 6 different marks.

Altogether

$$\binom{9}{1} + 5\binom{9}{2} + 10\binom{9}{3} + 10\binom{9}{4}$$
$$5\binom{9}{5} + \binom{9}{6} = 3003.$$

As for (a), once the marks are determined, the ordering is automatic, so the problem is really about finding how many ways there are to select a set of 6 numbers between 0 and 8.

2 Exponents and logarithms

1. (c) $a = 4, n = 6$ or $a = 8, n = 2$

2. (c) $x = \sqrt{\dfrac{1+\sqrt{5}}{2}}, y = \sqrt{\dfrac{\sqrt{5}-1}{2}}$

3. (a) $p = 9, q = \dfrac{9}{8}$

(b) $x = 12$

4. (a) 151
(b) 6
(c) 3
(d) 2

5. y^x

3 Polynomials

1. (a) $b \le -2$ or $b \ge 2$
(c) $b < -2$
(d)

b	*Number of real roots*	*sign*
$b > 2$	two	positive
$b = 2$	one	positive
$-2 < b < 2$	none	
$b = -2$	one	negative
$b < -2$	two	negative

2. (a) 11
(b) 23

3. $f(1002) = \dfrac{2}{1002}$

5. (b) $\dfrac{\sqrt{3}}{2}$

(c) $k = -\frac{3}{8}, \beta = \frac{7}{8}$ or $k = 0, \beta = 1$

4 Algebraic structures

1. (a) $x = \dfrac{1}{3}$
(b) line 2

2. (a) $x = 4$
(b) line 2

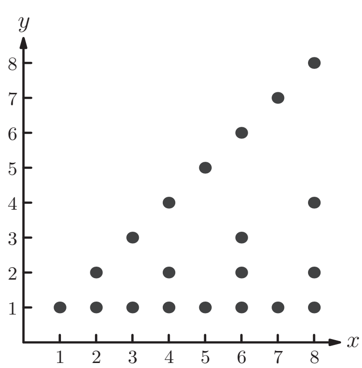
3. (b) $\Leftarrow$

4. (a) $x = -1$
(b) line 3

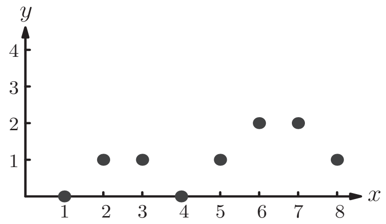
5. (a) $x = -2$
(b) line 2

5 The theory of functions

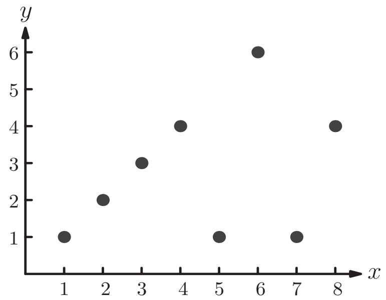
1. (a)



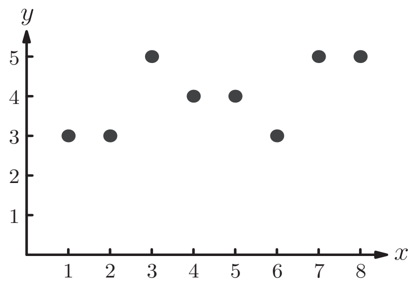
(b)



(c)



(d)



2. (a) $f(x) = kx$ for some $k \in \mathbb{R}$

(b) $f(x) = kx$ for some $k \in \mathbb{R}$

(c) $f(x) = x^2$

(d) $f(x) = x^2 + c$ for some $c \in \mathbb{R}$

3. $\frac{85}{2}$

4. $-\frac{20}{21}$

5. (a) No

(b) The range of $g(x)$ must be a subset of the domain of $f(x)$

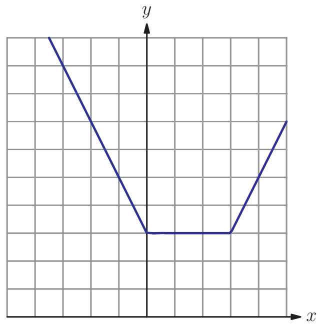
6. Not necessary: e.g. $f(x) = x^2, g(x) = x^3$

7. (a) $a \neq b$

(b) $b = 1, a \neq 1$

6 Transformations of graphs

1. (a)



(b) $-1, 4$

2. $-\frac{8}{3}, 1$

3. $-3, 3$

4. $y = |x - 3|$

5. $y = |x - z| + 1$

6. $y = |x - 1| + |x + 2|$

7. $y = ||x| - 1|$

7 Sequences and series

1. $x = \frac{1}{4}$

2. $-p - q$

3. (a) $\frac{7}{9}$

(b) $a \times 10^n - a$

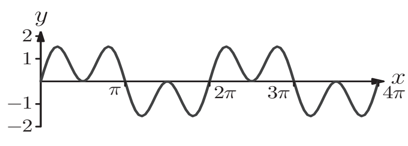
(c) $\frac{70(10^n - a)}{81} - \frac{7n}{9}$

8 Binomial expansion

All answers require proofs and are therefore not supplied as per the coursebook.

9 Circular measure and trigonometric functions

1 (a)



(b) 2π

2. (a) 2π

(b) π

(c) 2π

(d) π

3. $\frac{2\pi}{\gcd(m,n)}$

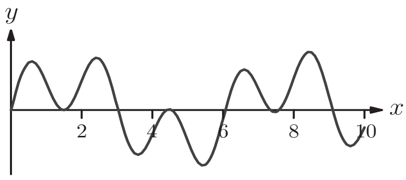
4. (a) 6π (b) 8π

(c) 12π (d) 24π

5. $2\pi \operatorname{lcm}(m,n)$

6. (a) 2

(b)



No

10 Trigonometric equations and identities

1. (a) $x = \frac{\pi}{9} + \frac{k\pi}{3}; \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}$

(b) $x = 1.2 + 2k\pi \ or \ \pi - 1.2 + 2k\pi; 1.2, \pi - 1.2$

(c) $x = \frac{\pi}{2} - 1.2 + 2k\pi$ or $\frac{\pi}{2} + 1.2 + 2k\pi; \frac{\pi}{2} - 1.2, \frac{\pi}{2} + 1.2$

(d) $x = \frac{5\pi}{4} + k\pi; \frac{\pi}{4}$

(e) $x = 2k\pi$ or $\frac{(2k+1)\pi}{3}; 0, \frac{\pi}{3}, \pi$

(f) $x = \frac{\pi}{4} + 2k\pi$ or $\frac{\pi}{12} + \frac{2k\pi}{3};$ $\frac{\pi}{12}, \frac{\pi}{4}, \frac{3\pi}{4}$

2. (a) 2, 3, 4 or 5

(b) 2, 3 or 4

3. 1

11a Geometry of triangles and circles

1. πp^2

2. (a) r

(c) (i) $\frac{\pi}{6}$ (ii) $r^2\left(\frac{\sqrt{3}}{4} + \frac{\pi}{6}\right)$

3. $(\pi - 2)r^2$

4. (a) $AM = BM = a$ because M lies on the circles with centres A and B and radius AB.
   (b) $30°$
   (c) $\frac{2}{3}\pi a$ (d) $a^2(\frac{\pi}{3}+1-\sqrt{3})$

11b Geometry of triangles and circles

1. (a) $CD = \frac{1+\sqrt{5}}{2}$
   (b) $\cos 36° = \frac{1+\sqrt{5}}{4}$
2. $a = 2R\sin\hat{A}$
3. 9820 m
4. It is not possible to find angles RAT and RBT because the points are not in the same plane; it is not true that, for example, $RAT = RAB - TAB$. So we cannot use the cosine rule in the last step of the calculation.

   They could measure angles of elevation of the mountain tops from A and B. We can then calculate the heights of the mountains and horizontal distance between the bases of the mountains, M and N, and then the distance between the peaks using the trapezium $MNRT$.
5. 1.52 au

12a Further trigonometry

1. $\frac{1}{8}$
4. (b) $\frac{\sqrt{6}}{2}$
5. $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{2\pi}{3}, \frac{4\pi}{3}$
7. (b) $\cos\theta(1-4\sin^2\theta)$
   (c) $\theta = 90°, 270°, 45°, 135°, 210°, 330°$

12b Further trigonometry

1. $x = \frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{3\pi}{4}$
2. 1 or 2
3. $\theta = \frac{\pi}{12}, \frac{5\pi}{12}$
4. $x = 180°, 225°$
5. $\theta = 285°, 345°$

6. (a) $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ (b) $\frac{1}{\sqrt{5}}$
7. (b) $\theta = 15, 195, 165, 345$
8. $x = \frac{3-\sqrt{6}}{6}$

13 Vectors

1. $\frac{1}{2}\boldsymbol{c} - \frac{1}{2}\boldsymbol{b} = \frac{1}{2}\overrightarrow{BC}$
2. (a) (i) $\frac{1}{2}\boldsymbol{c} - \boldsymbol{b}$
   (ii) $\frac{1}{3}\boldsymbol{c} - \frac{2}{3}\boldsymbol{b}$
   (b) (i) $\overrightarrow{CG} = \frac{1}{3}\boldsymbol{b} - \frac{2}{3}\boldsymbol{c}$
   (ii) $\overrightarrow{CF} = \frac{1}{2}\boldsymbol{b} - \boldsymbol{c}$
   (iii) $\overrightarrow{CG} = \frac{2}{3}\overrightarrow{CF}$
3. (b) $\boldsymbol{c} + \overrightarrow{CH}$
   (c)

$\overrightarrow{AH} \bullet (\boldsymbol{c}-\boldsymbol{b}) = \overrightarrow{AH} \bullet \boldsymbol{c} - \overrightarrow{AH} \bullet \boldsymbol{b}$	properties of scalar product
$= (\boldsymbol{b} + \overrightarrow{BH}) \bullet \boldsymbol{c} - (\boldsymbol{c} + \overrightarrow{CH}) \bullet \boldsymbol{b}$	from (b)
$= \boldsymbol{b} \bullet \boldsymbol{c} + \overrightarrow{BH} \bullet \boldsymbol{c} - \boldsymbol{c} \bullet \boldsymbol{b} - \overrightarrow{CH} \bullet \boldsymbol{b}$	properties of scalar product
$= \boldsymbol{b} \bullet \boldsymbol{c} - \boldsymbol{c} \bullet \boldsymbol{b}$	from (a)
$= 0$	

14 Lines and planes in space

1. (a) $\begin{pmatrix} 3t \\ t \end{pmatrix}$
   (b) $\begin{pmatrix} 5+kt \\ -2+4t \end{pmatrix}$
   (c) $-10 < k < 12$
   (d) $k = -4.5$
2. (b) $3b > a$
   (c) $a = \frac{3}{4}, b = \frac{7}{4}$
3. (a) $p = -1$
   (b) (ii) $a = -1$
   (c) Hint: Consider a plane determined by the intersecting lines.
4. Hint: Consider time as the third coordinate and the paths of snails as lines in three-dimensional space; then show that every pair of lines must intersect.

15 Complex numbers

1. Straight line, perpendicular bisector of the line segment between 1 and i
2. Circle, centre at $-\frac{1}{3}$ radius $\frac{2}{3}$
3. Circle, centre at $-\frac{1}{k^2-1}$, radius $\frac{k}{k^2-1}$ for $k \neq 1$. The limit case $k = 1$ produces the perpendicular bisector of the line segment between the orgin and 1.
4. Straight half-line through i, making angle $\frac{\pi}{3}$ with the horizontal. The graph should show a hollow dot at the origin.
5. The parts of the straight line through -1 and i outside the segment between them. Neither point is included.
6. The line segment between -1 and i. Neither point is included.
7. Semi-circle with the line segment between -1 and i as the diameter
8. Spiral
9. Real axis, excluding the orgin

16 Basic differentiation and its applications

2. $\sqrt[3]{\frac{1}{16}}$

ANSWER HINT

(2) Consider the requirement for two lines not meeting.

3. $(3,9), (5,25)$

ANSWER HINT

(3) Express j and k in terms of the sum or product of the x-coordinates of S and T.

4. a

5. (a) $B\left(-\frac{1}{a^3}, -a^3\right)$

   (b) $y + a^6 x = -2a^3$

6. (a) $Q\left(-\frac{1}{a}, -\frac{b}{a} + c\right)$

   (b) $a = \frac{b}{4(b^2+1)}$

   (c) $R = \left(-2\left(b + \frac{1}{b}\right), 2\left(b + \frac{1}{b}\right) + c\right)$

   (d) $b = \pm 2, a = \pm 0.1$

17 Basic integration and its applications

1. (a) 5 (b) 3

2. (b) $\frac{M(M-1)}{2}$

   (c) $\frac{\lfloor x \rfloor(\lfloor x \rfloor - 1)}{2} + \lfloor x \rfloor(\lfloor x \rfloor - x)$

3. (b) 1

4. (a) 4 (b) 4 (c) $\frac{2}{n}$

5. (a) (i) $\frac{16}{3}$ (ii) e – 1 (iii) 0

   (iv) $\frac{2}{\pi}$

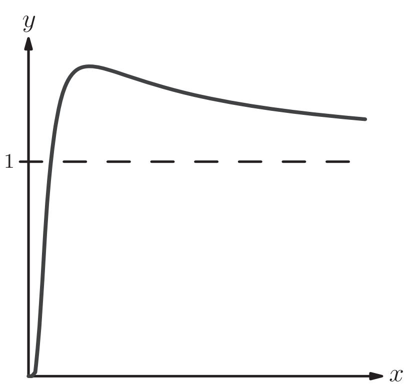
18 Further differentiation methods

1. (a) $y = -4e^x \sin x$

   (d) e.g. $f(x) = g(x) = x^3$

2. (a) $e^{\frac{1}{e}}$

   (b)



   (c) $x^{\frac{1}{x}-4}\left((1 - \ln x)^2 + x(2\ln x - 3)\right)$

19 Further integration methods

ANSWER HINT

(1) Make the substitution, rephrase with x as the variable and then add to the original equation.

(2) Write the integral from 0 to 1 and then apply the substitution and compare.

4. (b) $(e^{\pi/2}, 1)$

   (c) 1

20a Further applications of calculus

1. (b) (i) $m = \frac{r}{h}$ (ii) $\bar{x} = \frac{1}{4}h$

2. (a) $AB = \frac{a}{\sin\theta} + \frac{b}{\cos\theta}$

3. $u = \frac{2}{\sqrt{1-2x}}$

20b Further applications of calculus

1. (a) $y = x + \frac{x^2}{2} + c$

   (b) $y = -\cos x + c$

   (c) $y = \frac{x^3}{3} - \frac{x^2}{2} + c$

2. (a) $y = Ae^{-x}$

   (b) $y = A\sin(2x)$

   (c) $y = \sqrt{x(+c)}$

3. $y = ke^{-2x}$, $y = ke^{-3x}$

4. $\frac{d^2x}{dt^2} = -\frac{k}{m}x$, $x = a\sin\left(\sqrt{\frac{k}{m}}t\right)$,

   periodic (oscillating) with

   frequency $\sqrt{\frac{k}{m}}$

21 Summarising data

1. (a) $\mu = 2, \sigma^2 = \frac{2}{3}$

   (b) $\{1,2\}, \{1,3\}$ or $\{2,3\}$

   (c)

Sample	$\bar{x}$	s_n^2
1,2	1.5	0.25
1,3	2	1
2,3	2.5	0.25

   (d) 2; same as the population mean

   (e) 0.5; less than the population mean.

   (f) The mean sample mean remains 2, but the mean sample variance becomes $\frac{1}{3}$, which is $\frac{n-1}{n} = \frac{2-1}{2}$ of the population variance.

22 Probability

1. $\frac{\pi}{8}$

   Sketch a diagram and trace out the boundary where APB s a right angle, then determine the ratio of areas.

2. $\frac{7}{16}$

   Consider the arrival time of train A n a horizontal axis (0 to 20) and arrival of train B n a vertical axis (0 to 20). Calculate the area of the zone which corresponds to 'within 5 minutes of each other' as a fraction of the area of the whole square.

3. $\frac{13}{27}$

   The key fact which is not specified in the question is how the information given was chosen. If we assume (as intended) that one child was selected at random and its gender and birthday given, then by considering all 27 possibilities which could be valid out of the total 196, we find the answer given above. If instead there were some biased basis on which the child was selected, the answer would potentially be different; for example, if instead of randomly picking one of the pair, we deliberately picked the elder (but did not say so), the probability of both being boys would be 0.5; the gender and day of birth given would be irrelevant. If we chose an even more distorted basis, such as 'give details on a girl is there is one, otherwise on a boy, pick randomly if they are the same gender' then giving information on a boy would guarantee that the other was a boy.

4. (b) $\frac{1}{2}$ c) AAB

Consider the requirement for ABA to occur before AAB.

23 Discrete probability distributions

1. (a) For the mean to equal 1 when 1 is the lowest possible value for the distribution, it would have to have probability 1. But in that case the distribution would be constant and have zero variance.

Algebraically:

$$\mu = 1 = 1\times P(1) + 2\times P(2) + 3\times P(3) = P(1) + P(2) + P(3) + P(2) + 2\times P(3) = 1 + P(2) + 2\times P(3)$$

Since $P(2), P(3) \geq 0$ it follows that $P(2) = P(3) = 0$

(b) 2

This is harder to reason intuitively, but we can again solve the problem algebraically.

Let $p = P(1)$ and $q = P(y)$ so $P(0) = 1 - p - q$.

Then $p + yq = 1$and

$$p + y^2q - 1 = 1$$

$$\Rightarrow q(y^2 - y) = 1$$

$$\Rightarrow q = \frac{1}{y^2 - y}$$

In addition, we know that $p, q \geq 0$ and $p + q \leq 1$ so that $p \in [0, 1-q]$ nd so $yq = 1 - p \in [q, 1]$.

If we plot the graph of $q = \frac{1}{y^2 - y}$nd overlay the condition $y \in \left[1, \frac{1}{q}\right]$ we can find the minimum possible value for y is 2 with corresponding $q = 0.5$.

We can then check our solution, which has $P(0) = 0.5, P(1) = 0, P(2) = 0.5$

We also see that there is no upper limit on the value of y

2. (a) $\frac{481}{800}$

(b) $E(X) = 2.29, \mathrm{Var}(X) = 1.63$

raw out a probability tree for the three points to calculate these values.

3. (a) 0.1

(b) 0.32

(c) $\frac{3}{32}$

(d) $E(G) = 2.62, \mathrm{Var}(G) = 0.756$

(e) $E(B) = 2.6, \mathrm{Var}(B) = 0.86$

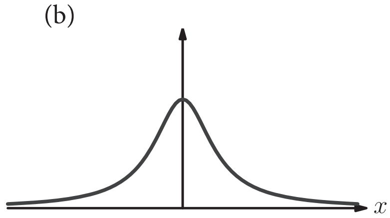
(f) $E(T) = 5.22, \mathrm{Var}(T) = 1.71$

24 Continuous distributions

1. (b) $\frac{2}{\pi}$

3. (a) π

(b)



(c) 0

(d) Infinite

Note that because the variance is infinite, the Mean value Theorem does not apply to the Cauchy distribution; the mean of n Cauchy variables does not tend towards a normal distribution as n gets larger.

4. (a) $\frac{1}{3}$

(b) $\frac{1}{9}$

(c) $1 - e^{-15}$

(d) $1 - e^{-15}$

A quality of the exponential distribution is that it 'restarts' without changing probabilities. That is, the probability of an event occurring in the first units (usually time or distance) is equal to the probability that, given no event *so far*, there will be an event in the next x units – irrespective of how far 'so far' represents. Examples in context are usually of waiting until a first event (for example the telephone ringing in an exchange) where events are distributed in a Poisson distribution along a timeline or in space. There will be a fixed probability that the telephone will next ring within 10 seconds. Whether or not it does, there will be the same probability that the telephone will ring in the subsequent 10 seconds, and so on. Each period is independent of the events in the previous one.