Self-assessment answers: 13 Vectors

1. (a) $\overrightarrow{\mathrm{MP}} = \overrightarrow{\mathrm{MB}} + \overrightarrow{\mathrm{BP}} = \frac{1}{3}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}$

$\overrightarrow{\mathrm{QN}} = \overrightarrow{\mathrm{QD}} + \overrightarrow{\mathrm{DN}} = \frac{1}{2}\boldsymbol{b} + \frac{2}{9}\boldsymbol{a}$

(b) $\overrightarrow{\mathrm{MP}} = \frac{1}{12}(4\boldsymbol{a} + 9\boldsymbol{b}),\ \overrightarrow{\mathrm{QN}} = \frac{1}{18}(4\boldsymbol{a} + 9\boldsymbol{b})$

$\therefore\ \overrightarrow{\mathrm{MP}} = \frac{3}{2}\overrightarrow{\mathrm{QN}}$ and so they are parallel. *[5 marks]*

2. (a) $\overrightarrow{\mathrm{BC}} = \begin{pmatrix} -10 \\ -7 \\ -1 \end{pmatrix}$ so $\mathrm{BC} = \sqrt{150} = 5\sqrt{6}$

(b) $\overrightarrow{\mathrm{CB}} = \begin{pmatrix} 10 \\ 7 \\ 1 \end{pmatrix}, \overrightarrow{\mathrm{CA}} = \begin{pmatrix} 4 \\ 6 \\ -1 \end{pmatrix}$

$\mathrm{A\hat{C}B} = \arccos\left(\frac{\overrightarrow{\mathrm{CB}} \cdot \overrightarrow{\mathrm{CA}}}{\mathrm{CB} \times \mathrm{CA}}\right)$

$= \arccos\left(\frac{81}{5\sqrt{6} \times \sqrt{53}}\right)$

$= 0.431$ radians ($24.7°$) *[6 marks]*

3. (a) $\overrightarrow{\mathrm{AC}} = \boldsymbol{a} + \boldsymbol{b}$

$\overrightarrow{\mathrm{BD}} = \boldsymbol{b} - \boldsymbol{a}$

(b) $\overrightarrow{\mathrm{AC}} \cdot \overrightarrow{\mathrm{BD}} = |\boldsymbol{b}|^2 - |\boldsymbol{a}|^2 = 0$

(c) Since the scalar product is zero, the two vectors (i.e. the diagonal directions) are perpendicular. *[7 marks]*

4. (a) $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} -6 \\ -2 \\ 5 \end{pmatrix}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 16 \\ -8 \\ 16 \end{pmatrix}$$

(b) Area of triangle ABC is given as $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = 4\left\|\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}\right\| = 4\sqrt{9} = 12$

(c) $\overrightarrow{AD} = \begin{pmatrix} 2 \\ p+1 \\ q-2 \end{pmatrix}$

Require that $\overrightarrow{AD} \cdot \overrightarrow{AB} = \overrightarrow{AD} \cdot \overrightarrow{AC} = 0$.

$\overrightarrow{AD} \cdot \overrightarrow{AB} = -4 + 2p + 2 + 3q - 6 = 2p + 3q - 8 = 0$ (1)

$\overrightarrow{AD} \cdot \overrightarrow{AC} = -12 - 2p - 2 + 5q - 10 = -2p + 5q - 24 = 0$ (2)

$(1) + (2) \Rightarrow 8q - 32 = 0$

$\Rightarrow q = 4$

$(1) \Rightarrow 2p + 4 = 0$

$\Rightarrow p = -2$

(d) Volume of tetrahedron ABCD is given by $\frac{1}{6}\left(\overrightarrow{AB} \times \overrightarrow{AC}\right) \cdot \overrightarrow{AD}$:

$$\overrightarrow{AD} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \text{so } \frac{1}{6}\left(\overrightarrow{AB} \times \overrightarrow{AC}\right) \cdot \overrightarrow{AD} = \frac{1}{6}\begin{pmatrix} 16 \\ -8 \\ 16 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \frac{72}{6} = 12$$

Proof of the above formula:

Volume of any (linearly) tapering solid = $\frac{1}{3}$ × base area × perpendicular height.

If D is considered the apex, then the height $h = \text{AD} \cos \theta$, where θ is the angle between AD and the perpendicular (normal to plane ABC).

So $h = \overrightarrow{AD} \cdot \hat{\boldsymbol{n}}$, where $\boldsymbol{n}$ is the normal to ABC.

But we already used the fact that $\frac{1}{2}\text{AB} \times \text{AC} = (\text{Area ABC})\,\hat{\boldsymbol{n}}$.

Therefore, $\frac{1}{3} \times$ base area $\times$ perpendicular height $= \frac{1}{3}\,(\text{Area ABC})\left(\hat{\boldsymbol{n}} \cdot \overrightarrow{\text{AD}}\right)$.

$$= \frac{1}{3}\left(\frac{1}{2}\overrightarrow{\text{AD}} \times \overrightarrow{\text{AC}}\right) \cdot \overrightarrow{\text{AD}}$$

$$= \frac{1}{6}\left(\overrightarrow{\text{AB}} \times \overrightarrow{\text{AC}}\right) \cdot \overrightarrow{\text{AD}}$$

[12 marks]