Supplementary sheet 1 The many applications of quadratic equations

An examination question on quadratic equations might look like this:

The sum of a number and the square of that number is 12. Find all possible values of this number.

This is a useful test to see whether you can solve quadratic equations, but it is not a very realistic problem. However, quadratic expressions appear in many naturally occurring situations. For example, the energy stored in a spring (E) is proportional to the force it is exerting (F) and to the extension (x):

$$E \propto Fx$$

But the force is also proportional to the extension:

$$F \propto x$$

Combining these two expressions we get:

$$E \propto x^2$$

Here are some more 'real world' quadratic expressions:

- Kinetic energy (E), mass (m) and velocity (v):

$$E = \frac{1}{2}mv^2$$

- Displacement (s), initial velocity (u), acceleration (a) and time (t) of a particle experiencing a constant acceleration:

$$s = ut + \frac{1}{2}at^2$$

- The Bernoulli equation for fluid flow:

$$p + \frac{1}{2}\rho V^2 + \rho gh \quad \text{is constant along a streamline}$$

  $p =$ pressure, $\rho =$ density of the fluid, $V =$ speed, $g =$ acceleration due to gravity and $h =$ height

- The 'logistic map' where x_n is the population of a species after n years. Each year due to natural birth and death rates the population increases by a factor of α whilst βx_n^2 die out due to lack of resources:

$$x_{n+1} = \alpha x_n - \beta x_n^2$$

In the following exercise, you will need to interpret the questions and decide which of the above equations to use.

Questions

1. A cricket ball has mass 0.3 kg.
   (a) What kinetic energy does the ball have when it is travelling at 10 ms^{-1}?
   (b) What speed would the ball have to travel at to have twice as much kinetic energy?

2. The final velocity of a car, v, under constant acceleration is given by the equation $v = u + at$.

A negative acceleration indicates that the car is decelerating.

(a) Show that the time taken for the car to stop is $-\frac{u}{a}$ once the brakes are applied.

(b) It takes 1 second for a driver to react to an event and apply the brakes. Find the stopping distance of a car travelling at 30 kmh^{-1} which can provide a braking acceleration of $-7\ \text{ms}^{-2}$.

(c) Find the stopping distance of the same car travelling at 60 kmh^{-1}.

3. A missile launched from ground level is designed to hit a target 15 m above the ground 180 m away. The horizontal and vertical components of its motion can be considered separately. Horizontally it has initial speed 60 ms^{-1} and has no acceleration. Vertically it has initial speed 20 ms^{-1} and has an acceleration due to gravity of $-10\ \text{ms}^{-2}$.

(a) Write down an expression for the displacement in the horizontal direction, x, in terms of time.

(b) Write down an expression for the displacement in the vertical direction, y, in terms of time.

(c) By eliminating time from these two equations show that the missile follows a parabolic path.

(d) Assuming that the ground is horizontal and that the missile does not hit anything find how far from the initial position the missile will land.

(e) Show that the missile does hit the target and find the time taken to do so.

(f) The initial horizontal speed is altered so that the target is hit two seconds earlier. Find the new initial horizontal speed.

4. The shape of an aeroplane wing is designed so air flows faster above the wing than below it. A particular plane has a wing of thickness 25 cm and is cruising with the airspeed immediately above the wing at 82 ms^{-1} and the airspeed below the wing at 66 ms^{-1}.

Air density is approximately 1.23 kgm^{-3} and acceleration due to gravity $g = 9.81\ \text{ms}^{-2}$.

(a) Assuming that the air travelling over and under the wing starts from the same streamline find the pressure difference between the top and bottom of the wing in Nm^{-2}.

(b) The two wings each have an area of $20\,\text{m}^2$. The lift force generated by the wings is given by

$$\text{lift} = \text{pressure difference} \times \text{area}.$$

Find the lift generated by the plane in N.

5. In a logistic model for the population of fish in a lake $\alpha = 1.4$ and $\beta = 0.0002$. Initially there are 1000 fish in the lake.

(a) How many fish are there after (i) 1 year? (ii) 5 years? (iii) 20 years?

(b) A stable population occurs when $x_{n+1} = x_n$. Prove that there are two stable populations and state their values.

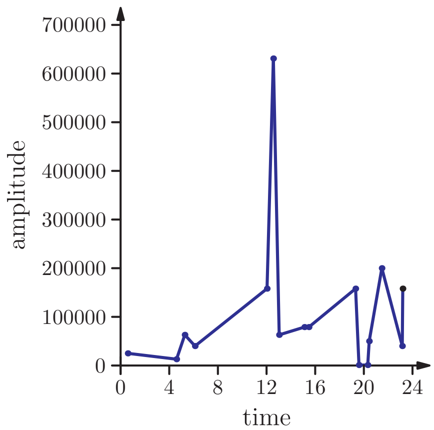
(c) Each year, k fish are removed from the lake. By considering the quadratic equation $x = \alpha x - \beta x^2 - k$ find the largest number of fish that can be removed and still leave a non-zero stable population.

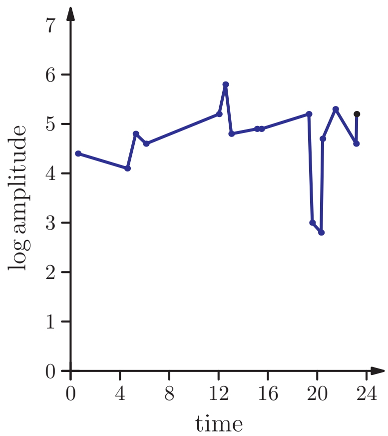
Models like these are actually used to determine the amount of fish which can safely be caught without causing extinction.

Supplementary sheet 2 Logarithmic scales and log-log graphs

Some physical quantities cover a large range of values and this can make them difficult to work with and to represent graphically.

For example, the magnitude of an earthquake can be measured by the amplitude of the oscillation shown on the seismograph. But the amplitudes of earthquakes vary a lot. The graph on the left shows the amplitudes of earthquakes recorded on 16th December 2011. The horizontal axis shows time (hours and minutes). The strongest earthquake had the amplitude about 100 times larger than the weakest one, and this makes it difficult to show them on the same graph.



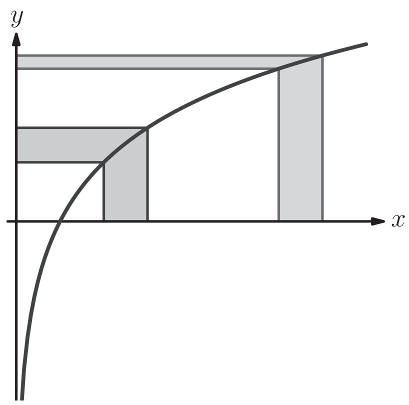


(Source: US Geological Survey, http://neic.usgs.gov/neis/qed/)

The graph on the right shows the same data but using a *logarithmic scale*. This is where the values on the y-axis are obtained by taking the logarithm (in base 10) of the original data. This scale for measuring the strength of earthquakes is called the Richter scale.

The magnitude of an earthquake on a Richter scale is defined by $M = \log\left(\frac{A}{k}\right)$ where A is the amplitude of the earthquake and k is a constant (which depends on the distance of the measuring station from the epicentre).

A logarithmic scale makes the distances between large numbers appear smaller, because of the shape of the logarithm graph.



Here are some other examples of logarithmic scales:

- In chemistry, pH measures the acidity of a solution. For a water-based solution, it is defined as $-\log[H^+]$, where $[H^+]$ is the concentration of the hydrogen ions measured in moles per litre of solution. Pure water has pH equal to 7.

- The strength of a sound signal in decibel (dB) is $10\log\left(\frac{P}{P_0}\right)$ where P is the power of the sound signal and P_0 is the power of the reference sound signal, corresponding to the sound pressure of $20\mu\text{Pa}$ (this is 20×10^{-6} Pa, and Pascal is the unit of pressure).

  It can also be defined in terms of the sound pressure, which is proportional to the square root of the power. In that case the strength in decibel is $20\log\left(\frac{S}{S_0}\right)$ where S is the sound pressure and $S_0 = 20\mu\text{Pa}$.
- Some of our senses operate on a logarithmic scale; for example, we perceive equal ratios in frequencies as equal differences in pitch. The frequencies of successive notes in a twelve-tone octave differ by a constant factor of $\sqrt[12]{2} \approx 1.06$, so that two tones which are an octave apart have frequencies differing by a factor of 2.

Studies of children in an Amazonian tribe show that they perceive numbers on a logarithmic scale, so that they think of the difference between 1 and 2 as larger than the difference between 11 and 12.

Questions

1. How many times larger is the amplitude of an earthquake of magnitude 6.5 on the Richter scale than an earthquake of magnitude 5.2?

2. Two earthquakes' Richter scale magnitudes differ by 1. What can you say about their magnitudes?

3. One earthquake has amplitude 3 times larger than another one. What is the difference between their Richter scale magnitudes?

4. Solutions with a higher concentration of hydrogen ions than water are called acids.

(a) Is the pH of acids higher or lower than 7?

(b) Sea water contains one mole of hydrogen ions per 1.3×10^8 litres. Find its pH.

5. Black coffee has a pH of 5.2 and tomato juice has a pH of 4.5. Which solution has a higher concentration of hydrogen ions? How many times higher is it?

6. What is the sound pressure of a signal whose strength is 1dB ?

7. (a) If the sound pressure is doubled, by how many decibels does the strength change?

(b) If the power of a sound signal is doubled, by how many decibels does the strength change?

8. Hearing damage in humans occurs when exposed to the sound pressure of around $0.356\,\text{Pa}$ over a long period of time. How many decibels is this?

9. The music note middle C has frequency $261.6\,\text{Hz}$. Which note has frequency 370 Hz? (The English names for musical notes are C, C#, D, D#, E, F, F#, G, G#, A, A#, B.)

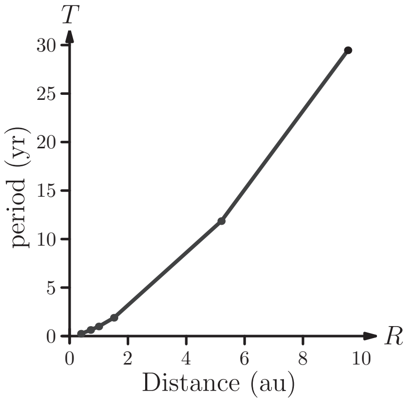
Log-log graphs

Sometimes it is useful to take a logarithm of the quantities on both axes of a graph. This is called a *log-log graph*. For example, if two quantities are related by a power law, $y = kx^p$, then the logarithms of the two quantities satisfy $\log y = \log k + p \log x$, which means that the graph of $\log y$ against $\log x$ is a straight line. This is particularly useful if we are trying to find an equation to model a set of data. It is difficult to deduce the power p from the shape of the original graph, but on the log-log graph p is the gradient of the straight line, which is much easier to find.

Questions

10. The table and graph below show the period of revolution around the Sun (T) and the average distance from the Sun (R) for six planets of the Solar system. The period is measured in (Earth) years and the distance in astronomical units (1 au is the average distance of the Earth from the Sun).

Planet	Distance R (au)	Period T (years)
Mercury	0.39	0.24
Venus	0.72	0.62
Earth	1.00	1.00
Mars	1.52	1.88
Jupiter	5.20	11.86
Saturn	9.54	29.46



(a) Plot a log-log graph of the data.

(b) Use your calculator to find the line of best fit in the form $y = a + bx$, where $y = \log T$ and $x = \log R$.

(c) Hence find constants k and p so that $T = kR^p$.

> The last equation is known as Kepler's 3rd law.

Supplementary sheet 3 A history of logarithms

Logarithms were originally used as a calculation tool. The idea was first developed by the Scottish mathematician John Napier in the early 17th century.

Because of the rule:

$$\log(ab) = \log a + \log b$$

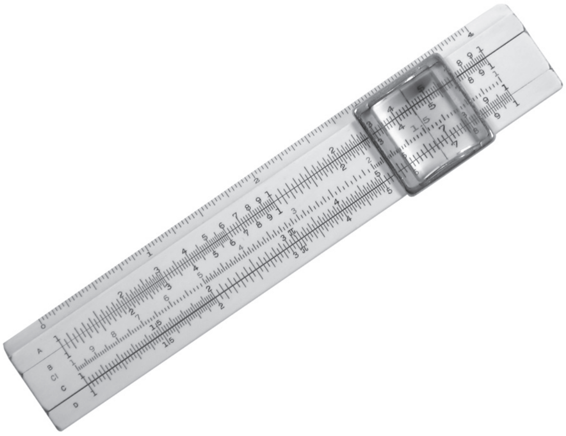
logarithms turn multiplication into addition, which is much easier to perform without a calculator. In this worksheet you will find out how logarithms were used in calculations.

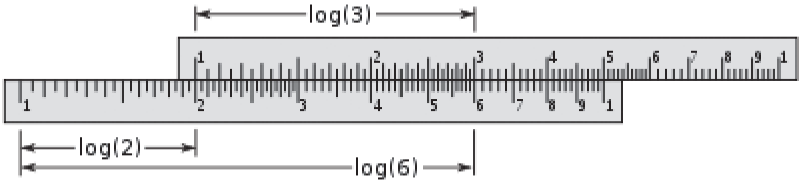
Were logarithms invented or discovered?

Values of logarithms were published in booklets called log tables. An extract from a log table is shown below. For example, the third number in the second row is $\log(1.12) = 0.0492$.

+	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.0	.0000	.0043	.0086	.0128	.0170	.0212	.0253	.0294	.0334	.0374
1.1	.0414	.0453	.0492	.0531	.0569	.0607	.0645	.0682	.0719	.0755
1.2	.0792	.0828	.0864	.0899	.0934	.0969	.1004	.1038	.1072	.1106
1.3	.1139	.1173	.1206	.1239	.1271	.1303	.1335	.1367	.1399	.1430
1.4	.1461	.1492	.1523	.1553	.1584	.1614	.1644	.1673	.1703	.1732
1.5	.1761	.1790	.1818	.1847	.1875	.1903	.1931	.1959	.1987	.2014
1.6	.2041	.2068	.2095	.2122	.2148	.2175	.2201	.2227	.2253	.2279
1.7	.2304	.2330	.2355	.2380	.2405	.2430	.2455	.2480	.2504	.2529
1.8	.2553	.2577	.2601	.2625	.2648	.2672	.2695	.2718	.2742	.2765
1.9	.2788	.2810	.2833	.2856	.2878	.2900	.2923	.2945	.2967	.2989
2.0	.3010	.3032	.3054	.3075	.3096	.3118	.3139	.3160	.3181	.3201
2.1	.3222	.3243	.3263	.3284	.3304	.3324	.3345	.3365	.3385	.3404
2.2	.3424	.3444	.3464	.3483	.3502	.3522	.3541	.3560	.3579	.3598
2.3	.3617	.3636	.3655	.3674	.3692	.3711	.3729	.3747	.3766	.3784
2.4	.3802	.3820	.3838	.3856	.3874	.3892	.3909	.3927	.3945	.3962
2.5	.3979	.3997	.4014	.4031	.4048	.4065	.4082	.4099	.4116	.4133
2.6	.4150	.4166	.4183	.4200	.4216	.4232	.4249	.4265	.4281	.4298
2.7	.4314	.4330	.4346	.4362	.4378	.4393	.4409	.4425	.4440	.4456
2.8	.4472	.4487	.4502	.4518	.4533	.4548	.4564	.4579	.4594	.4609
2.9	.4624	.4639	.4654	.4669	.4683	.4698	.4713	.4728	.4742	.4757
3.0	.4771	.4786	.4800	.4814	.4829	.4843	.4857	.4871	.4886	.4900
3.1	.4914	.4928	.4942	.4955	.4969	.4983	.4997	.5011	.5024	.5038

A slide rule is a mechanical device which uses a logarithmic scale to perform multiplication and division. They were widely used until calculators became available in the 1970s. The second picture shows the calculation $2 \times 3 = 6$ on a slide rule.





Questions

1. For $x = 1.25$ and $y = 2.30$ use the table to calculate $\log x + \log y$. Hence find the value of xy correct to one decimal place.
2. Use the log tables to calculate $3.17 \div 1.25$ correct to one decimal place.
3. Use log tables to calculate 1.15^7.
4. Find the value of $\sqrt[3]{3.17}$ correct to two decimal places.
5. Evaluate 18.3×112 using the log table (you will need a bit of ingenuity and some rules of logarithms!). Check the answer using your calculator.

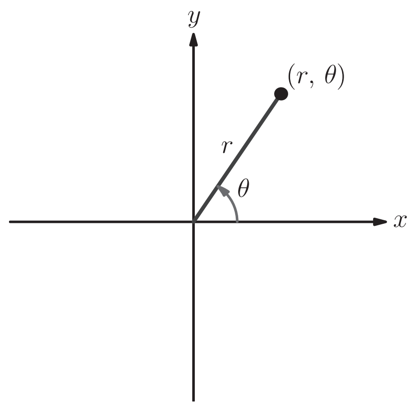
Supplementary sheet 4 Coordinate systems and graphs

You are familiar with the Cartesian coordinate system to describe positions of points and write equations of graphs. In this system, the position of a point is determined by its horizontal and vertical distances from the origin O (x- and y-coordinates) and graphs are described by equations connecting the two coordinates. So, for example, the equation $y = x$ represents all points for which the two coordinates are the same, and they form a diagonal line.

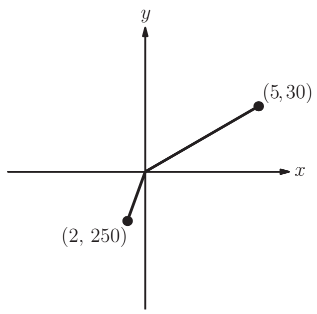
But there are other ways of describing positions of points. You are already familiar with bearings, where the position is described by the distance from the origin and the angle measured clockwise from the North. This method of describing positions is used in navigation.

For examples of bearings see Prior learning Section U on the CD-ROM.

A similar coordinate system also used in mathematics is called polar coordinates. In this system, the position of a point is described by the distance from the origin and the angle measured anti-clockwise from the positive x-axis. The coordinates of the point are usually written as (r,θ), and here we will measure the angle θ in degrees.



The diagram below shows the points with coordinates $(5,30)$ and $(2, 250)$



Questions

1. On the same diagram, plot the points with these polar coordinates.

(a) $(2,60)$

(b) $(5,120)$

(c) $(3,270)$

(d) $(5,0)$

2. Sketch the line or curve consisting of the points satisfying the following equations:

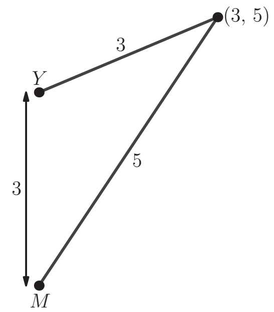
(a) $r = 3$

(b) $\theta = 60$

(c) $r = \theta$

Polar coordinates are often used when studying closed curves, for example the orbits of planets. There are several three-dimensional versions of polar coordinates, the most common ones being cylindrical polars and spherical polars.

All the coordinate systems described above measure the position relative to a single **origin**. This may reflect our culture, where we tend to consider ourselves to be 'at the centre'. But we can think of other coordinate systems that would be more appropriate for different cultures. For example, certain Maori tribes in New Zealand always refer to both 'you and me' when describing relative positions. They may say 'The basket is 20 steps from you and 15 steps from me'. So for them it might make more sense to use a coordinate system with two origins, Y and M, and describe the position of a point by the two distances from those origins. In this coordinate system the point shown on the diagram would have coordinates $(3,5)$.



Have you ever thought about how your language and culture affects your mathematics? Is mathematics a 'universal' language?

3. Plot the points which in a two-origin coordinate system (with the origins 3 units apart) have the following coordinates (y,m):

(a) $(4,4)$

(b) $(4,1)$

(c) $(0,3)$

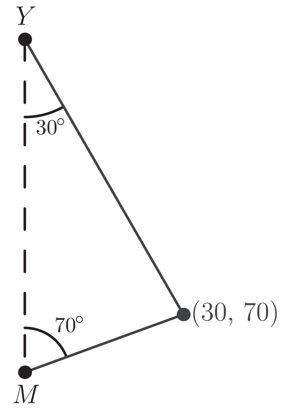
4. Sketch the line or curve consisting of the points satisfying the following equations:

(a) $y = 1$

(b) $m = 3$

(c) $y = m$

Another way that we could describe positions of points in the Maori coordinate system would be to use angles instead of distances. For example, if we measure angles in degrees from the line connecting the two origins, then the point shown on the diagram has coordinates $(\theta,\phi)=(30,70)$.



5. Plot the points which in the above coordinate system (with the origins 3 units apart) have the following coordinates (θ,ϕ):

(a) $(50,20)$

(b) $(40,40)$

(c) $(30,120)$

6. Sketch the line or curve consisting of the points satisfying the following equations:

(a) $\theta=60$

(b) $\phi=150$

(c) $\theta=\phi$

(d) $\theta+\phi=90$

You will see in chapter 11 how this last way of describing positions can be used with trigonometry to calculate distances that cannot be measured directly, for example when surveying land.

Supplementary sheet 5 Long-term behaviour of sequences and series

You met geometric series in chapter 7 and saw that their behaviour depends on the value of the common ratio. The sum of the series either increases without a limit or approaches a limiting value (S_∞). In this worksheet we will investigate the behavior of other sequences and series as more and more terms are added. You may wish to use a spreadsheet to help with this investigation.

Questions

1. A sequence is defined by the formula $u_n = \dfrac{n+1}{n}$.

(a) Write down the first five terms of the sequence.

(b) Investigate what happens to u_n as n increases.

(c) By writing the formula for u_n in a different form, explain the result found in (b).

2. The *logistic map* is defined by $x_{n+1} = kx_n(1-x_n)$ where k is a constant. The behaviour of the sequence depends on the value of k and on the starting value x_1. We will only look at the values of k between 2 and 4 and the values of x_1 between 0 and 1, but you may also wish to investigate other values.

An application of this equation to model population growth was explored on Supplementary sheet 1 'The many applications of a quadratic equation' on the CD-ROM.

(a) Find the first 20 terms of the sequence for the following values of k and x_1:

(i) $k = 2.5$, $x_1 = 0.3$

(ii) $k = 2.5$, $x_1 = 0.5$

(iii) $k = 2.5$, $x_1 = 0.8$

(b) Find the first 20 terms of the sequence when:

(i) $k = 2.8$, $x_1 = 0.3$

(ii) $k = 2.8$, $x_1 = 0.5$

(iii) $k = 2.8$, $x_1 = 0.8$

EXAM HINT

Find more terms if you are not sure what happens to the sequence.

Describe the long-term behaviour of the sequence for the values of k tested above.

(c) Generate the sequence with $k = 3.1$ for several starting values x_1 between 0 and 1.

How would you describe the long-term behaviour of the sequence?

(d) Investigate the long-term behaviour of the sequence when:

(i) $k = 3.5$

(ii) $k = 4$

You may need many more terms (around 100 should make the pattern clear), and graphical representation may help.

What are the differences between the two cases? Does the behaviour of the sequence change when you change the starting value x_1?

> The behaviour observed when $k = 4$ is called *chaotic*; the sequence never repeats and small changes in the initial value lead to very different sequences. This type of behaviour was first observed in the late 19th century but, as it requires the use of computers, it has only been studied in more detail since the 1960s. It can be used to explain unpredictable behaviour in the motion of satellites (the three-body problem), weather patterns and population growth.

3. In this question we will investigate long-term behaviour of series.

(a) Investigate what happens as you add more and more terms to the series

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$$

Try to find out what the limiting value is; the answer may surprise you!

(b) The *harmonic series* is

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$$

Does this series approach a limiting value as you add more terms?

(c) Investigate the behaviour of the series

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots$$

for different values of the power p.

> This is called the Riemann Zeta Function. It can be linked beautifully to prime numbers, and is the source of one of the most famous unsolved problems in mathematics: The Riemann Hypothesis.

Supplementary sheet 6 The Chess legend and extreme numbers

When the inventor of the game of chess showed it to the ruler of India, he was so impressed that he invited the inventor to name his reward. The inventor asked that one grain of rice was put on the first square of the chessboard and then double the number of grains for each subsequent square. The ruler was offended that the inventor had asked for such a small reward, but quickly realized that he was very wrong!

Questions

1. (a) How many grains of rice would need to be placed on the 64th square of the chessboard?

(b) How many grains in total should the inventor receive?

> This amount of rice would form a heap larger than Mount Everest, and correspond to around 1000 times the world production of rice in 2010.

2. Many people have very poor intuition when dealing with very large or very small numbers.

Estimate the following ratios:

(a) $\dfrac{\text{Distance from Sun to Earth}}{\text{Distance from Moon to Earth}}$

(b) $\dfrac{\text{Width of Helium atom}}{\text{Width of Helium nucleus}}$

(c) $\dfrac{\text{Width of human hair}}{\text{Width of human liver cell}}$

(d) $\dfrac{\text{Volume of a swimming pool}}{\text{Volume of a can of soft drink}}$

(e) $\dfrac{\text{Weight of an elephant}}{\text{Weight of a mouse}}$

(f) $\dfrac{\text{Distance from Sun to the nearest star}}{\text{Distance from Sun to Earth}}$

(g) $\dfrac{\text{Number of atoms in the observable universe}}{\text{Number of atoms in an inflated balloon}}$

(h) $\dfrac{\text{Speed of light}}{\text{Speed of Usain Bolt}}$

Supplementary sheet 7 Babylonian multiplication

The Babylonians lived in the area currently known as Iraq from the 3rd millennium BCE to the 1st millennium BCE. They made many advances in mathematics, astronomy, medicine and philosophy.

The Babylonians did not do multiplication directly. They used a method derived from multiplying out brackets instead. This was mainly due to the fact that they worked in base 60. This means they had single symbols for all of the numbers from 0 to 59, and then the digits to the left count how many sixties are in the number. For example, 151 would be written as 2 '60s' and 31 'units'. We could say that it is (2,31) in base 60.

Questions

1. Write the following decimal numbers in base 60:

(a) 35

(b) 99

(c) 312

(d) 1000

(e) 3601

In decimal multiplication we often 'remember' the results for multiplying two digits together, for example $8\times 6=48$. For more complicated multiplication we then combine results together; so for example $38\times 6=30\times 6+8\times 6=180+48=228$.

If we could not remember this we would have to do repeated addition:

$$38\times 6=\underbrace{38+38+38}_{\text{100 plus14 left over}}+\underbrace{38+38+38}_{\text{100 plus14 left over}}=100+100+14+14=228$$

In base 60, both 38 and 6 are digits:

$$38\times 6=\underbrace{38+38}_{\text{60 plus 16 left over}}+\underbrace{38+38}_{\text{60 plus 16 left over}}+\underbrace{38+38}_{\text{60 plus 16 left over}}=60+60+60+16+16+16=(3,48)$$

2. Use repeated addition to work out the following digit multiplications in base 60, without using the results from the decimal system.

(a) 42×5

(b) 18×8

(c) 13×15

In the decimal system the following digit multiplications need to be learnt:

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

However, if you remember that $0\times x=0$, $1\times x=x$ and $x\times y=y\times x$ all that we need to remember is the shaded area which is 36 results.

3. How many results must be remembered for all of the digit multiplications in base 60?

The result of the previous question should convince you that our decimal method of learning all the 'digit products' is not a good choice for base 60. The Babylonians came up with an ingenious method which meant that they only had to learn the 'digit-squares'. There are 60 such digit-squares to remember.

x	*x* squared		*x*	*x* squared		*x*	*x* squared		*x*	*x* squared		*x*	*x* squared		*x*	*x* squared	
	60's	units		60's	units		60's	units		60's	units		60's	units		60's	units
0	0	0	**10**	1	40	**20**	6	40	**30**	15	0	**40**	26	40	**50**	41	40
1	0	1	**11**	2	1	**21**	7	21	**31**	16	1	**41**	28	1	**51**	43	21
2	0	4	**12**	2	24	**22**	8	4	**32**	17	4	**42**	29	24	**52**	45	4
3	0	9	**13**	2	49	**23**	8	49	**33**	18	9	**43**	30	49	**53**	46	49
4	0	16	**14**	3	16	**24**	9	36	**34**	19	16	**44**	32	16	**54**	48	36
5	0	25	**15**	3	45	**25**	10	25	**35**	20	25	**45**	33	45	**55**	50	25
6	0	36	**16**	4	16	**26**	11	16	**36**	21	36	**46**	35	16	**56**	52	16
7	0	49	**17**	4	49	**27**	12	9	**37**	22	49	**47**	36	49	**57**	54	9
8	1	4	**18**	5	24	**28**	13	4	**38**	24	4	**48**	38	24	**58**	56	4
9	1	21	**19**	6	1	**29**	14	1	**39**	25	21	**49**	40	1	**59**	58	1

The trick was to find an expression for xy involving only square numbers.

4. Show that

$$xy=\frac{1}{2}\left((x+y)^2-x^2-y^2\right)$$

This was the formula that the Babylonians used. For example

$$38\times 6=\frac{1}{2}\left(44^2-38^2-6^2\right)$$
$$=\frac{1}{2}\left((32\text{ lots of }60+16\text{ units})-(24\text{ lots of }60+4\text{ units})-(36\text{ units})\right)$$
$$=\frac{1}{2}(8\text{ lots of }60-24\text{ units})$$
$$=4\text{ lots of }60-12\text{ units}$$
$$=3\text{ lots of }60+48$$

This is the same result that we found using repeated addition.

5. Find the following products using the Babylonian method.

(a) 31×23

(b) 14×19

(c) 25×30

Supplementary sheet 8 How does your calculator work out sin and cos?

Before calculators (which wasn't so long ago!) people had to use tables to find the values of sin, cos and tan. Most textbooks contained tables at the back, or you could get special booklets of tables. Here is an extract from such a table:

Trigonometric sines of angles

Angle (Deg)	Sine	Angle (Deg)	Sine	Angle (Deg)	Sine	Angle (Deg)	Sine
0.0	0.0000	24.0	0.4067	48.0	0.7431	72.0	0.9511
0.5	0.0087	24.5	0.4147	48.5	0.7490	72.5	0.9537
1.0	0.0175	25.0	0.4226	49.0	0.7547	73.0	0.9563
1.5	0.0262	25.5	0.4305	49.5	0.7604	73.5	0.9588
2.0	0.0349	26.0	0.4384	50.0	0.7660	74.0	0.9613
2.5	0.0436	26.5	0.4462	50.5	0.7716	74.5	0.9636

But how were these numbers calculated? And how does your calculator work out the values of trigonometric functions?

Throughout history mathematicians have used various ways of calculating sin, cos and tan. Initially the methods involved triangles and chords of circles, but in the 14th century Indian mathematician-astronomer Madhava of Sangamagrama discovered what is now known as *Taylor series*. This method was further developed in 17th and 18th century Europe, and is now used by calculators and computers to find values of many complicated functions. The method is based on the fact that many functions can be approximated by a polynomial (an expression involving natural powers of x only).

In the following questions you will use your calculator to investigate how the Taylor series method works for sine and cosine functions.

Questions

1. Sketch the graph of $y = \sin x$, where x is in radians, for $-\pi \le x \le \pi$.

   On the same graph sketch $y = x$. Zoom in around the origin. What do you notice?

2. Add the graph of $y = x - \frac{1}{6}x^3$. Zoom out back to $-\pi \le x \le \pi$. What do you notice now?

3. Change the window to $-2\pi \le x \le 2\pi$ and draw the graph of $y = \sin x$.

   (a) Add the graph of $y = x$.

   (b) Replace it by $y = x - \frac{1}{6}x^3$.

   (c) Replace it by $y = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$.

   What do you notice?

   The Taylor series for $\sin x$ is $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$.

   Investigate what happens if you add more and more terms.

4. The Taylor series for $\cos x$ is $1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots$.

Draw the graph of $y=\cos x$ for $-2\pi \le x \le 2\pi$.

(a) Add the graph of $y=1-\frac{1}{2}x^2$.

(b) Replace it by the graph $y=1-\frac{1}{2}x^2+\frac{1}{24}x^4$.

(c) Try adding more terms of the Taylor series. What do you notice?

5. Find the percentage errors when $\sin(1)$, $\sin(5)$, $\cos(1)$ and $\cos(5)$ are approximated by the Taylor series with:

(a) two terms

(b) three terms

(c) four terms

You have already met the idea of approximation with binomial expansion in chapter 8 of the coursebook.

Supplementary sheet 9 Applications of differentiation

An examination question will often give you a function $f(x)$ and ask you to find its stationary points, classifying them as maxima, minima or points of inflexion.

We know that to do this we solve:

$$f'(x) = 0$$

and then use the second derivative test at the stationary point x_0:

$$f''(x_0) > 0 \Rightarrow x_0 \textit{ is a local minimum}$$
$$f''(x_0) < 0 \Rightarrow x_0 \textit{ is a local maximum.}$$

If $f''(x_0) = 0$ we cannot be sure of the nature of the stationary point and so need to consider the gradient either side.

We also need to remember that the above procedure gives the stationary points, but the largest or smallest value of the function might be at an end point of the domain.

But where can this be applied in real life? Well, the answer is in many different areas of mechanics, engineering, physics and economics (to name but a few). Here we look at some examples in the context of economics, focusing on a number of different functions that economists use to predict the behaviour of individuals or businesses.

Utility function

One such key function is a **utility function**, $u(x)$, of an individual. This is a function that evaluates an individual's satisfaction at consuming a certain amount of an item x.

Questions

1. A consumer has the utility function

$$u(x) = \frac{\ln x}{x}$$

for consuming chocolate bars (x). Given there are no restrictions on his purchasing and that he wishes to maximise his utility, how much chocolate will he choose to consume?

When you learn how to differentiate quotients in chapter 18 of the coursebook, you will be able to find the exact answer without using GDC.

2. A consumer wishes to buy some T-shirts (x) and pairs of shorts (y).

His utility function for these two products is

$$u(x, y) = 6x^2 + 5y$$

but this time his purchasing is constrained, as he only has £34 to spend. If T-shirts cost £8 each and shorts each cost £10,

(a) write down his budget constraint

(b) find how many of each good he should choose to buy to maximise his utility.

3. A consumer wishes to buy some computer games (x) and some DVDs (y). His utility function for these two products is:

$$u(x,y)=2x^2+5y$$

and he has £155 to spend. If computer games each cost £40 and DVDs each cost £25,

(a) write down his budget constraint

(b) find how many of each goods he should choose to buy to maximise his utility.

4. Robinson Crusoe is marooned on a desert island. The only foods available to him are coconuts (x) and fish (y). He has no real preference for consuming these and has found his utility function to be:

$$u(x,y)=\sqrt{x}\sqrt{y}.$$

However, he has to gather the food before he can eat it and has found that the amount of time it takes him to gather coconuts is:

$$t_c=x^2$$

but fishing is more time consuming, taking him:

$$t_f=4y^2.$$

All this is tiring and Robinson needs 8 hours of rest, leaving him 16 hours to gather the food.

(a) Write down an equation for his time constraint.

(b) Hence find how many coconuts and how much fish he should choose to collect and eat.

Optimisation in business

Where a business is concerned, some important functions to consider include the:

- **cost function**, $c(x)$, which gives the cost of manufacturing x amount of the product
- **revenue function**, $r(x)$, which gives the amount of money the business will receive for selling x amount of the product
- **profit function**, $p(x)=r(x)-c(x)$, which gives the overall profit calculated from revenue minus costs.

5. A manufacturing firm that makes bicycles in Cambridge has found that its cost function, $c(x)$, is:

$$c(x)=53x^2-540x-335$$

where x is the number of bicycles produced.

(a) Find the number of bikes they would produce if only interested in minimising their production costs.

The amount of money the company receives for selling the bikes, the revenue function, $r(x)$, is given by:

$$r(x)=-x^3+128x^2+1260x+660.$$

(b) Find the number of bikes the company would produce if only interested in maximising their revenue.

(c) Write down the profit function, $p(x)$, and hence find the number of bikes the company will actually produce.

Instead of considering the cost function, a firm might instead use a function that gives the extra cost incurred by producing one more unit of the product, as, in practice, this is often easier to deduce. This function is known as the **marginal cost function**, $M_c(x)$. As we know from our understanding of differentiation, this function is the rate of change of the cost with x, so this marginal cost function is simply the derivative of the cost function.

Similarly, we can define the **marginal revenue**, $M_r(x)$, and **marginal profit**, $M_p(x)$, functions, i.e.

- $M_c(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}c(x)$
- $M_r(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}r(x)$
- $M_p(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}p(x)$.

Therefore, to maximise profit we just set the marginal profit to be 0 (and of course check that this gives a maximum):

$$M_p(x) = 0 \Rightarrow M_r(x) - M_c(x) = 0 \Rightarrow M_r(x) = M_c(x).$$

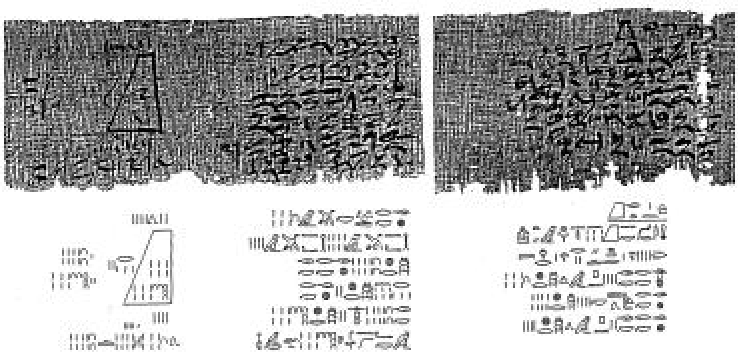
So the optimal production point will be where marginal revenue is equal to marginal cost. This is the same as we did in question 5, but it does put a slightly different perspective on the calculation. It is now saying that a firm will continue to produce until the point where the extra revenue gained from producing one more unit of the product is the same as the extra cost incurred.

It might seem unnecessary to talk of marginal costs and revenues, when we can find the optimal production point by considering just the actual cost and revenue functions. However, economists frequently use the notion of marginal cost and revenue functions, for example in the analysis of price-setting behaviour and how this varies between a completely competitive market and a monopoly.

Supplementary sheet 10 Who invented calculus?

Calculus is the study of infinitesimal objects, usually via differential calculus (infinitely small changes in the value of a function) or integral calculus (the sum of infinitely small rectangles). Within the Western world it is thought that calculus was invented in the late 17th century by either **Gottfried Leibniz** (1646–1716) or **Isaac Newton** (1642–1727). There was considerable debate at the time about which of the two actually discovered it with Leibniz publishing first but Newton making reference to his 'method of fluxions' many years previously. However, this argument may be irrelevant as there is evidence that the methods of calculus had been developed in other countries many centuries, or in some cases millennia, earlier.

Ancient Egyptians



The Moscow Papyrus.

This is a text written in Egypt in roughly 1850 BC.

> If you are told: a truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top: You are to square the 4; result 16. You are to double 4; result 8. You are to square the 2; result 4. You are to add the 16 and the 8 and the 4; result 28. You are to take 1/3 of 6; result 2. You are to take 28 twice; result 56. See, it is of 56. You will find (it) right.
>
> Source: http://en.wikipedia.org/wiki/Moscow_Mathematical_Papyrus, as given in Gunn & Peet, *Journal of Egyptian Archaeology*, 1929, 15: 176.

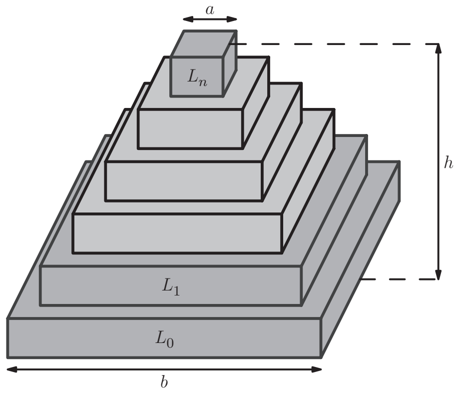
This suggests that the Egyptians knew the formula for a truncated square pyramid with top width a, bottom width b and height h was:

$$V = \frac{1}{3}h(a^2 + ab + b^2).$$

Given the limited algebra that was available at the time, it is believed that they came to this result via an approximation method that is effectively integral calculus.

Questions

1. Suppose the truncated pyramid is approximated by $n+1$ layers of cuboids, labelled $L_0, L_1, \ldots, L_n$.



(a) What is the vertical thickness of each cuboid?

(b) Show that the width of the rth cuboid from the bottom is given by $b-\frac{r}{n}(b-a)$. (Note: the bottom cuboid is the 0th and the top one is the nth).

(c) Find an expression for the volume of the rth cuboid and write it in the form $h(A-Br+Cr^2)$.

(d) The following results were known to the Egyptians:

$$\sum_1^n 1=n,\ \sum_1^n r=\frac{n(n+1)}{2},\ \sum_1^n r^2=\frac{n(n+1)(2n+1)}{6}.$$

Using these results find an expression for the total volume of the layers of cuboids. (Remember that there are n + 1 layers, not n.)

(e) Explain why when n gets very large this expression will get closer and closer to the formula for the volume of a truncated pyramid.

Liu Hui and the Nine Chapters

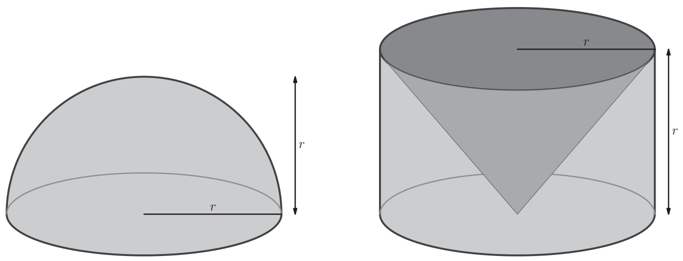
In 263 CE the Chinese mathematician **Liu Hui** published notes on the Nine Chapters of the Mathematical art, a summary of the mathematics of several generations of scholars. Within it he applied Cavalieri's Principle:

If two shapes have the same cross sectional area at every height, then they have the same volume.

This splitting up of shapes into shapes of a lower dimension is an important precursor of integration.

Questions

2. In this question we shall consider the following diagrams to find a formula for the volume of a hemisphere:



You may assume that the volume of a cone is $\frac{1}{3}\pi r^2 h$, the volume of a cylinder is $\pi r^2 h$ and the area of a circle is πr^2.

(a) Show that at a height y above the base the radius of the hemisphere is $\sqrt{r^2 - y^2}$ and hence find the cross sectional area at this height.

(b) Find the radius of the cone at a height y above its tip and hence find its cross sectional area at this height.

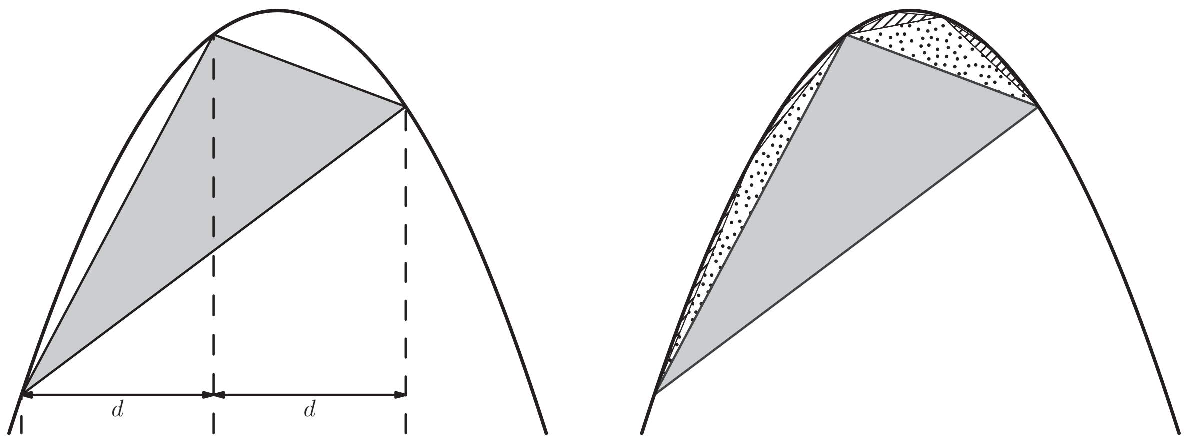
(c) Find the cross sectional area of the ring that is outside the cone but within the cylinder at height y.

(d) Use Cavalieri's principle to find the volume of the hemisphere.

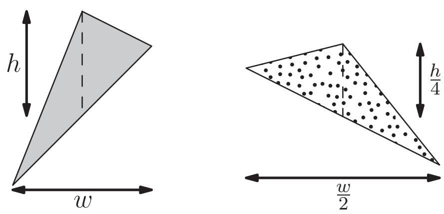
Archimedes and the quadrature of the parabola

The above result was also shown by **Archimedes** (287–212 BCE), and he was so proud of it that he had it engraved upon his tombstone.

Archimedes also found a method for finding the area under a parabola. He split the area up into triangles, with a vertex of each triangle being positioned horizontally halfway between the other two vertices. He continued this process to fill the remaining area.



Because of the way it was constructed, the dotted triangles have half of the width of the shaded triangle. Because it is within a parabola, it also has one quarter of the height.



Therefore, it has one eighth of the area.

Questions

3. What is the area of the striped triangle, in the second diagram above, relative to the shaded triangle?

4. How many dotted triangles are there? How many striped triangles are there?

5. Show that the total area may be given by a geometric series and find its sum. Hence show that the area under the parabola is $\frac{4}{3}$ of the area of the blue triangle.

6. Using integration show that this result is true for the chord from $(-1,0)$ to $(1,0)$ on the curve $y = 1 - x^2$.

Notable others

Antiphon (480–411 BCE) and later **Eudoxus** (408–355 BCE) are generally credited with implementing the method of exhaustion, which made it possible to compute the area and volume of regions and solids by breaking them up into an infinite number of recognisable shapes.

Aryabhata (476–550) was an Indian scholar and astronomer who published a major work in 499 CE showing that for small changes in an angle, δx

$$\delta(\sin x) = \delta x \cos x.$$

Alhazen (965–1040) was an Iraqi mathematician who came up with a formula for the volume of a paraboloid (a three dimensional version of a parabola), and in the process found formula for integrals of polynomials up to order four.

The French mathematician **Nicole Oresme** (1320–1382) along with colleagues at Oxford University proved the 'Merton mean speed theorem': that a uniformly accelerated body travels the same distance as a body with uniform speed whose speed is half the final velocity of the accelerated body.

René Descartes (1596–1650) made the crucial link between algebra and geometry that still bears his name, cartesian coordinates, which allowed Newton and Leibniz to formalise calculus.

French mathematician **Pierre de Fermat** (1601–1665) found a formula for integrating integral powers of x and also used the gradients of tangents to find minimum and maximum values of a function.

English mathematician **Isaac Barrow** (1630–1677) considered the tangent to a curve to be the limit of chords through the curve.

Having read through all this you may wonder what Newton and Leibniz did that has given them the status of the 'inventors of calculus'. Their main contribution was the fundamental theorem of calculus – the idea that differentiation and integration are the opposite of each other.

1. Was calculus invented or discovered?
2. How many of the mathematicians mentioned here did you know? Of those, how many are European? Why is it that some mathematicians are more famous than others?
3. Was the creation of calculus a revolutionary or an evolutionary knowledge shift?

Supplementary sheet 11 Measuring risk: what is the most dangerous way to travel?

Fatality statistics

The following data shows the numbers of deaths in different modes of transport in the US in 2005:

Mode of transport	Number of deaths
Pedestrian	4 881
Bicycle	784
Motorbike	4 398
Car	18 440
Passenger airline	66
Train	2

Source: www.medicine.ox.ac.uk/bandolier/booth/Risk/trasnsportpop.html

Questions

1. How reliable are these data?
   (a) How much will it change from year to year?
   (b) Are the numbers likely to be underestimates or overestimates?
   (c) What are the likely sources of error?

2. Using this data, what can you say about the relative danger of travelling by car, by bicycle and by passenger airplane?
   (a) What is the most dangerous way to travel?
   (b) What are the problems with using this data to measure risk?

Relative risk I

One issue with the above data is that it is not a fair comparison. There are far fewer deaths on the train than in a car, but one reason for that is that there is much more travel by car than by train. To make a fair comparison we need to take this into account. One way of doing this is dividing the number of deaths by the number of miles travelled by passengers using each method. The following figures show the number of deaths per billion passenger kilometres in the UK in 1999.

Mode of transport	Deaths per billion passenger km
Air	0.02
Rail	0.9
Water	0.3
Car	2.8
Two-wheeled motor vehicle	112
Pedal cycle	41
Pedestrian	49

Source: http://plus.maths.org/content/opinion-2

3. What are the advantages and disadvantages of using deaths per person per km travelled as a measure of the risk?
4. Using this data, what can you say about the relative danger of travelling by car, by bicycle and by passenger airplane?
5. This data comes from the UK in 1999. Does this mean that it cannot be compared to the US in 2005?

Relative risk II

An alternative measure of risk is the number of fatalities per hour. Below are the fatalities per million exposure hours in the USA:

Activity	Fatalities per million exposure hours
Skydiving	128.71
General flying	15.58
Motorcycling	8.80
Scuba diving	1.98
Living	1.53
Swimming	1.07
Snowmobiling	.88
Motoring	.47
Water skiing	.28
Bicycling	.26
Airline flying	.15
Hunting	.08

Source: www.exponent.com/

6. What are the advantages and disadvantages of using deaths per hour as a measure of the risk?
7. Using these data, what can you say about the relative danger of travelling by car, by bicycle and by passenger airplane?
8. Suggest why 'living' is more dangerous than water skiing.
9. Use this table and the table of 'Deaths per billion passenger km' to estimate the average speed of a car and a bicycle. (These figures are going to give very rough estimates; only to within an order of magnitude.) Do these figures seem reasonable?
10. Explain why cycling appears much safer using this measure than the 'Deaths per km' measure.

Relative risk III

Below is an estimate for the number of deaths per trip in the US in 1995:

Mode	Deaths per 100 million trips
Car	9
Plane	3
Bicyclists	26
Pedestrians	29

Source: www.vtpi.org/puchertq.pdf

11. What are the advantages and disadvantages of using deaths per trip as a measure of the risk?

12. Using the data, what can you say about the relative danger of travelling by car, by bicycle and by passenger airplane?

13. Which of the following claims can be substantiated by at least one of the sets of measures of risk:

    (a) Cycling is safer than driving.

    (b) Driving is safer than cycling.

    (c) Flying is 140 times safer than driving.

    (d) Flying is three times safer than driving.

    (e) The train is the safest form of transport.

    (f) The plane is the safest form of transport.

14. Based upon all of the evidence, what is your opinion about the relative risks of each form of transport?

Supplementary sheet 12 Significant discoveries

In 2011 there was much media coverage of the experiments conducted at the Large Hadron Collider in Geneva. Particular attention was paid to the evidence for the existence of the Higgs boson. This is a theoretical particle that could explain how the universe was formed. Towards the end of the year it was reported that the data gathered were not yet robust enough to claim a conclusive discovery.

In particle physics many particles are not observed directly, because they exist for only a tiny fraction of a second. What is observed is the particles that they decay into. Theories suggest that when the Higgs boson decays there will be more 'Tau' particles observed than normal. The problem is that there are always some Tau particles produced. We therefore need a cut off at which we say that there are significantly more Tau particles produced than we would normally expect. In particle physics this is called the five-sigma limit.

Questions

1. If a variable follows a normal distribution, what is the probability of it being more than five standard deviations away from the mean? This is how unlikely an event has to be according to the current theory before a new theory is accepted in particle physics.

2. What probability of being wrong would you accept when sentencing someone to life imprisonment?

3. What probability of being wrong would you accept when deciding on the best temperature for baking a cake?

4. The probability of two individuals having the same genetic marker in a DNA fingerprint is about 7.5%. A DNA fingerprint compares 13 such genetic markers.

(a) Assuming that the genetic markers are independent, find the probability of two randomly chosen individuals sharing all 13 markers.

(b) Find the probability of two randomly chosen individuals sharing at least 9 out of 13 genetic markers.

(c) In Arizona, there is a database of 65 000 felons. How many pairs would be expected to share at least 9 out of 13 genetic markers.

> This question is based upon the findings of Kathryn Troyer, a crime lab analyst in Arizona. At the time, the FBI claimed that the chances of two people matching at 9 out of 13 markers was one in one billion. Do you agree?

In real life we can never be 100% certain about anything. We must choose a reasonable level of doubt that we can accept. This varies depending upon the situation, and it is called the **significance level**. If you study the statistics option (Option 7) you will learn much more about this.

Answers: Supplementary sheets

1 The many applications of quadratic equations

1 (a) 15 J
  (b) 14.1 ms^{-1}

2 (b) 13.3 m
  (c) 36.5 m

3. (a) $x = 60t$
  (b) $y = 20t - 5t^2$
  (d) 240 m
  (e) 3 s
  (f) 180 ms^{-1}

4. (a) 1460 N/m^2
  (b) 58400 N

5. (a) (i) 1200
    (ii) 1800
    (iii) 2000
  (b) 0, 2000
  (c) $k = \dfrac{(1-\alpha)^2}{4\beta} = 200$

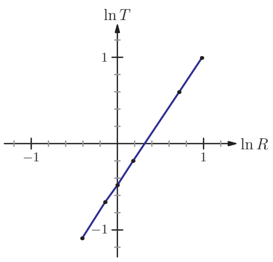
2 Logarithmic scales and log-log graphs

1. 20.0
2. Differ by a factor of 10
3. $\log 3 = 0.477$

4 (a) Lower
  (b) 8.11

5. Tomato juice – 5 times higher
6. 22.4μPa
7. (a) $20\log 2 = 6.02$
  (b) $10\log 2 = 3.01$

8. 85 dB
9. F#
10. (a)



(b) $a \approx 0$, $b \approx 1.5$

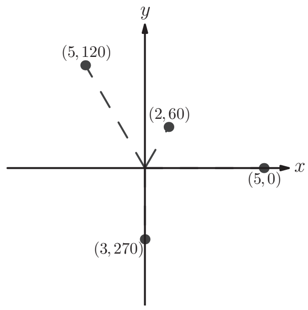
(c) $k = 1, p = \dfrac{3}{2}$

3 A history of logarithms

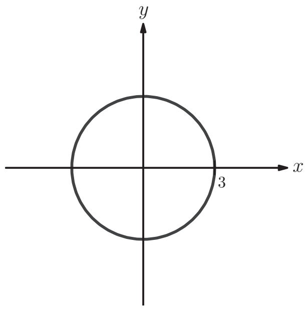
1. 2.9
2. 2.5
3. 2.66
4. 1.47
5. 2050

4 Coordinate systems and graphs

1.



2. (a)



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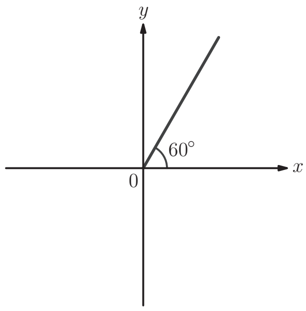

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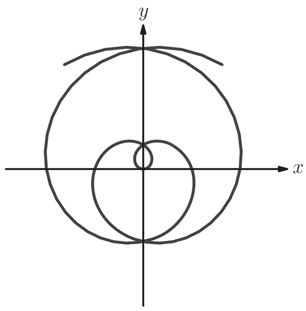

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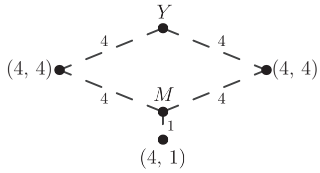
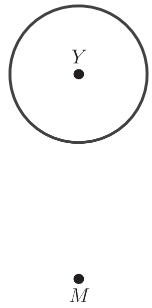


4. (a)



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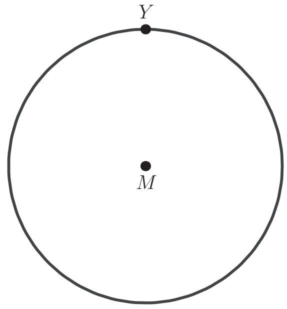
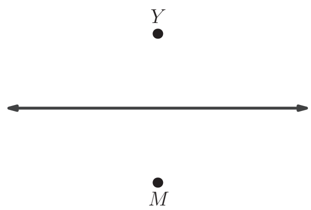
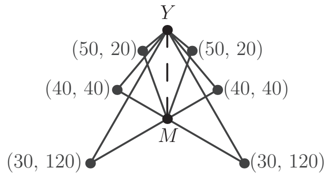


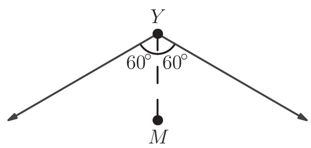
(c)



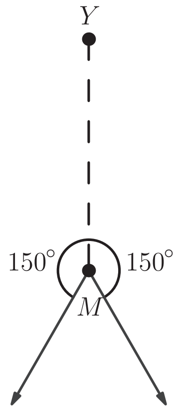
5.



6. (a)



(b)



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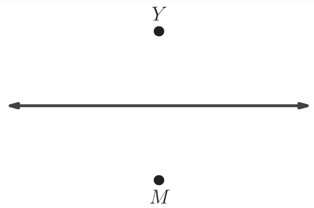
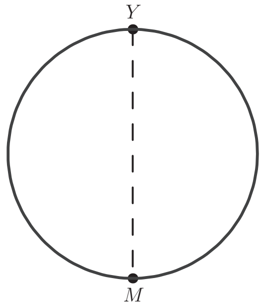


(d)



5 Long-term behaviour of sequences and series

1. (a) $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}$

(b) Gets closer to 1

(c) $u_n = 1 + \frac{1}{n}$

3. (a) approaches $\frac{\pi^2}{6}$

(b) No

6 The chess legend and extreme numbers

1. (a) 2^{63}

(b) $2^{64} - 1 \approx 1.84 \times 10^{19}$

2. (a) 400

(b) 100 000

(c) 2

(d) 1 000 000

(e) 100 000

(f) 25 000 000

(g) 10^{55}

(h) 30 000 000

7 Babylonian multiplication

1. (a) $(0,35)$

(b) $(1,39)$

(c) $(5,12)$

(d) $(16,40)$

(e) $(1,0,1)$

2. (a) $(3,30)$

(b) $(2,24)$

(c) $(3,15)$

3. 1711

5. (a) $(11,53)$

(b) $(4,26)$

(c) $(12,30)$

8 How does your calculator work out sin and cos?

5.

	2 terms	3 terms	4 terms
$\sin 1$	0.967	0.0233	0.000325
$\sin 5$	1550	1160	452
$\cos 1$	7.46	0.253	0.00454
$\cos 5$	4150	5030	2620

9 Applications of differentiation

1. About 2.7 chocolate bars

2. (a) $8x + 10y \le 34$

(b) Four T-shirts and no shorts

3. (a) $40x + 25y \le 155$
$\Rightarrow 8x + 5y \le 31;$

(b) 0 computer games and 6 DVDs

4. (a) $x^2 + 4y^2 \le 16$

(b) $x = 2.82, y = 1.41$

5. (a) $5.1 \approx 5$ bikes

(b) $x = 90$ bikes

(c) $p(x)$
$= -x^3 + 75x^2 + 1800x + 995$
$x = 60$ bikes

10 Who invented calculus?

1. (a) $\frac{h}{n+1}$

(c) $h\left(\frac{b^2}{n+1} - \frac{2b(b-a)}{n(n+1)}r + \frac{(b-a)^2}{n^2(n+1)}r^2\right)$

(d) $h\left(b^2 - b(b-a) + \frac{2n+1}{6n}(b-a)^2\right)$

(e) Because $\frac{2n+1}{6n} \to \frac{1}{3}$

2. (a) $\pi(r^2 - y^2)$

(b) y

(c) $\pi r^2 - \pi y^2$

(d) $\frac{2}{3}\pi r^3$

3. $\frac{1}{64}$

4. 2 green, 4 yellow

11 Measuring risk: what is the most dangerous way to travel?

The questions on this sheet are meant as ideas for discussion and many of them do not have 'right' or 'wrong' answers.

12 Significant discoveries

1. 6×10^{-7}

4. (a) 2×10^{-15}

(b) 4×10^{-8}

(c) 86

You should multiply the probability by the number of pairs, not felons!