Revision answers: Calculus (Topic 6)

Coursebook chapters: 16–20

1. $f'(x) = 3ax^2 + 2bx + 4$

$f''(x) = 6ax + 2b$

So,

$f'(2) = 0 \quad \Rightarrow \quad 12a + 4b + 4 = 0 \quad \Rightarrow \quad 3a + b = -1$

$f''(2) = 10 \quad \Rightarrow \quad 12a + 2b = 10 \quad \Rightarrow \quad 6a + b = 5$

Solving simultaneously gives $a = 2$, $b = -7$ *[5 marks]*

2. $\int_{-\pi/3}^{\pi/2} \cos\left(\frac{1}{2}x\right)\mathrm{d}x = \left[2\sin\left(\frac{1}{2}x\right)\right]_{-\pi/3}^{\pi/2} = 2\left(\frac{\sqrt{2}}{2} - \left(-\frac{1}{2}\right)\right) = \sqrt{2} + 1$ *[4 marks]*

3. (a) $\frac{(x-2)(x-6)}{\sqrt{x}} = \frac{x^2 - 8x + 12}{x^{\frac{1}{2}}} = x^{\frac{3}{2}} - 8x^{\frac{1}{2}} + 12x^{-\frac{1}{2}}$

i.e. $a = \frac{3}{2}, b = \frac{1}{2}, c = -\frac{1}{2}$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}x^{\frac{1}{2}} - 4x^{-\frac{1}{2}} - 6x^{-\frac{3}{2}}$

At $x = 4, \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{4}$

$\therefore$ gradient of normal $= -4$

When $x = 4$, $y = -2$.

So, $y - (-2) = -4(x - 4) \quad \Rightarrow \quad y = -4x + 14$

(c) (i) At the intersection with the x-axis, $y = 0$: $0 = -4x + 14 \Rightarrow x = \frac{7}{2}$.

At the intersection with the y-axis, $x = 0$: $y = 14$.

So, $P\left(\frac{7}{2}, 0\right)$, $Q(0, 14)$

(ii) Area $= \frac{1}{2} \times \frac{7}{2} \times 14 = \frac{49}{2}$ *[13 marks]*

4. From GDC, the intersection point is (0.7628, 0.9137) and the second curve has root 1.0327.

If $a = 0.7628$ and $b = 1.0327$, the area shown is $\int_0^a x^{\frac{1}{3}}\,\mathrm{d}x + \int_a^b \left(\mathrm{e}^{-x^2} - x\right)\mathrm{d}x$.

Area = 0.5227 + 0.1207 = 0.643 (3SF) *[6 marks]*

5. (a) $y' = 12x^3 - 24x^2 + 12x$

At stationary points $y' = 0$

$\Rightarrow x^3 - 2x^2 + x = 0 \Rightarrow x(x-1)^2 = 0 \Rightarrow x = 0, 1$

So stationary points are $(0, -2)$ and $(1, -1)$

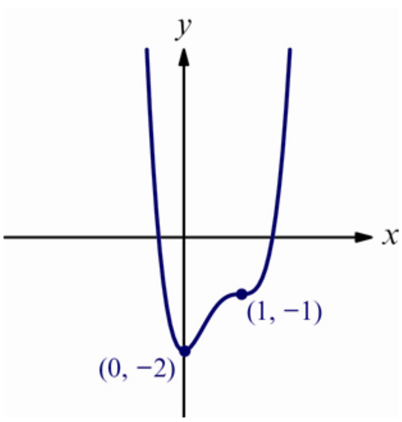
(b) $y'' = 36x^2 - 48x + 12$

At $x = 0$, $y'' = 12 > 0 \Rightarrow (0, -2)$ local minimum.

At $x = 1$, $y'' = 0$ so check y' either side of $x = 1$.

When $x = \frac{1}{2}$, $y' > 0$ and when $x = \frac{3}{2}$, $y' > 0 \Rightarrow (1, -1)$ (positive) point of inflexion.

(c)



[12 marks]

6. Using the reverse chain rule or the substitution, $u = x^2 - 9$:

$$\int \frac{4x}{x^2-9}\,\mathrm{d}x = 2\ln\left(x^2-9\right)+c = \ln\left(x^2-9\right)^2 + c$$ *[6 marks]*

7. Using the product rule for (a) and quotient rule for (b):

(a) $y' = 2\mathrm{e}^{2x}\tan^2 3x + \mathrm{e}^{2x} \times 2(\tan 3x) \times 3\sec^2 3x$

$= 2\mathrm{e}^{2x}\tan^2 3x + 6\mathrm{e}^{2x}\tan 3x \sec^2 3x$

$= 2\mathrm{e}^{2x}\tan 3x(\tan 3x + 3\sec^2 3x)$

(b) $$y' = \frac{\frac{2x}{1+x^2}\left(1+x^2\right) - \ln\left(1+x^2\right)2x}{\left(1+x^2\right)^2}$$

$$= \frac{2x\left(1-\ln\left(1+x^2\right)\right)}{\left(1+x^2\right)^2}$$ *[9 marks]*

8. (a) (i) $v = \int a\,\mathrm{d}t = 3t^2 - 22t + c$

$v = 35$ when $t = 0 \Rightarrow c = 35$

$\therefore\ v = 3t^2 - 22t + 35$

(ii) $3t^2 - 22t + 35 = 0 \Rightarrow (3t-7)(t-5) = 0 \Rightarrow t = \frac{7}{3}$, 5 seconds

(iii) For local max/min, $v' = 0$, so $6t - 22 = 0 \Rightarrow t = \frac{11}{3}$.

However, at $t = \frac{11}{3}$, $v = -\frac{16}{3}$, whereas at $t = 0$, $v = 35$.

$\therefore$ max speed $= 35\ \mathrm{ms}^{-1}$

(b) We know that P has negative velocity for $\frac{7}{3} < t < 5$, so P is moving back towards O here.

So find the displacement for $0 < t < \frac{7}{3}$ and $\frac{7}{3} < t < 5$ separately:

$$s_1 = \int_0^{7/3} v\,\mathrm{d}t = \int_0^{7/3} 3t^2 - 22t + 35\,\mathrm{d}t = \left[t^3 - 11t^2 + 35t\right]_0^{7/3} = \frac{931}{27}\text{ m}$$

$$s_2 = \int_{7/3}^{5} v\,\mathrm{d}t = \int_{7/3}^{5} 3t^2 - 22t + 35\,\mathrm{d}t = \left[t^3 - 11t^2 + 35t\right]_{7/3}^{5} = -\frac{256}{27}\text{ m}$$

$$\therefore s = \frac{931}{27} + \frac{256}{27} = \frac{1197}{27} \approx 44\text{ m}$$

[11 marks]

9. (a) (i) $y = 3^x \Rightarrow \ln y = \ln(3^x) \Rightarrow \ln y = x\ln 3$

Differentiating: $\frac{1}{y}y' = \ln 3 \Rightarrow y' = y\ln 3 = 3^x \ln 3$

(ii) $\int 3x\,\mathrm{d}x = \frac{3^x}{\ln 3} + c$

(b) $u = 3^x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 3^x \ln 3 = u\ln 3$

When $x = 0$, $u = 1$ and when $x = 1$, $u = 3$.

$$\therefore \int_0^1 \frac{3^{2x}}{3^x+1}\mathrm{d}x = \int_1^3 \frac{u^2}{u+1}\frac{\mathrm{d}u}{u\ln 3} = \frac{1}{\ln 3}\int_1^3 \frac{u}{u+1}\mathrm{d}u$$

$$= \frac{1}{\ln 3}\int_1^3 1 - \frac{1}{u+1}\mathrm{d}u$$

$$= \frac{1}{\ln 3}\left[u - \ln(u+1)\right]_1^3 = \frac{2-\ln 2}{\ln 3}$$

[12 marks]

10. $2x + 2yy' - 3y - 3xy' = 0 \;\Rightarrow\; y' = \dfrac{3y - 2x}{2y - 3x}$

$y' = 0 \;\Rightarrow\; 3y - 2x = 0 \;\Rightarrow y = \dfrac{2x}{3}$

Substituting into the original function:

$$x^2 + \left(\frac{2x}{3}\right)^2 - 3x\left(\frac{2x}{3}\right) + 20 = 0 \;\Rightarrow\; x^2 = 36 \;\Rightarrow\; x = \pm 6$$

So the stationary points are (6, 4) and (−6, −4). *[9 marks]*

11. (a) $\cos 2x = 1 - 2\sin^2 x \;\Rightarrow\; \sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$\therefore \int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x)\,dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + c = \frac{1}{4}(2x - \sin 2x) + c$$

(b) $V = \pi \int_0^{\pi/2} y^2 \, dx = \pi \int_0^{\pi/2} x \sin^2 x \, dx$

$$= \pi\left(\left[\frac{x}{4}(2x - \sin 2x)\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{1}{4}(2x - \sin 2x)\,dx\right)$$

$$= \pi\left(\left[\frac{x}{4}(2x - \sin 2x)\right]_0^{\pi/2} - \left[\frac{1}{4}\left(x^2 + \frac{1}{2}\cos 2x\right)\right]_0^{\pi/2}\right)$$

$$= \pi\left(\frac{\pi^2 + 4}{16}\right)$$

[12 marks]

12. (a) Height of triangular face $= \sqrt{3}$

Area of triangular face of water of depth h and base b, $A = \frac{1}{2}bh$

By similar triangles: $\dfrac{h}{b} = \dfrac{\sqrt{3}}{2} \;\Rightarrow\; b = \dfrac{2h}{\sqrt{3}}$

So, the volume of water at height h, $V = \dfrac{1}{2}bh \times 6 = \dfrac{1}{2} \times \dfrac{2h}{\sqrt{3}} h \times 6 = 2\sqrt{3}h^2$.

(b) Capacity of tank = $6\sqrt{3}$, so when quarter full $V = \frac{3}{2}\sqrt{3}$.

$$\frac{3}{2}\sqrt{3} = 2\sqrt{3}h^2 \quad \Rightarrow \quad h = \frac{\sqrt{3}}{2}$$

Then, $\dfrac{\mathrm{d}V}{\mathrm{d}h} = 4\sqrt{3}h$

And $\dfrac{\mathrm{d}h}{\mathrm{d}t} = \dfrac{\mathrm{d}h}{\mathrm{d}V}\dfrac{\mathrm{d}V}{\mathrm{d}t} = \dfrac{1}{4\sqrt{3}h} \times \dfrac{600}{10^6} = \dfrac{3}{20000\sqrt{3}h}$

So, when $h = \dfrac{\sqrt{3}}{2}$, $\dfrac{\mathrm{d}h}{\mathrm{d}t} = \dfrac{3}{20000\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)} = \dfrac{1}{10000}\ \mathrm{ms}^{-1}\left(\text{or } \dfrac{1}{100}\ \mathrm{cm}^{-1}\right)$ *[7 marks]*

13. (a) $y = \arctan\dfrac{x}{3} \quad \Rightarrow \quad \tan y = \dfrac{x}{3}$

$$\therefore\ \sec^2 y y' = \frac{1}{3} \quad \Rightarrow \quad y' = \frac{1}{3\sec^2 y} = \frac{1}{3\left(1+\tan^2 y\right)} = \frac{1}{3\left(1+\frac{x^2}{9}\right)} = \frac{3}{x^2+9}$$

(b) $\displaystyle\int_{\sqrt{3}}^{3} \arctan\frac{x}{3}\,\mathrm{d}x = \int_{\sqrt{3}}^{3} 1\times\arctan\frac{x}{3}\,\mathrm{d}x$

$$= \left[x\arctan\frac{x}{3}\right]_{\sqrt{3}}^{3} - \int_{\sqrt{3}}^{3}\frac{3x}{x^2+9}\,\mathrm{d}x$$

$$= \left[x\arctan\frac{x}{3}\right]_{\sqrt{3}}^{3} - \left[\frac{3}{2}\ln\left(x^2+9\right)\right]_{\sqrt{3}}^{3} = \frac{\left(9-2\sqrt{3}\right)\pi - 18\ln\frac{3}{2}}{12}$$ *[9 marks]*

14. (a) $11 + 10x - x^2 = 11 - (x-5)^2 + 25 = 36 - (x-5)^2$

i.e. $a = 36$, $b = 5$

(b) $\displaystyle\int_5^c \frac{1}{\sqrt{11+10x-x^2}}\,\mathrm{d}x = \frac{\pi}{6} \quad \Rightarrow \quad \int_5^c \frac{1}{\sqrt{36-(x-5)^2}}\,\mathrm{d}x = \frac{\pi}{6} \quad \Rightarrow \quad \left[\arcsin\left(\frac{x-5}{6}\right)\right]_5^c = \frac{\pi}{6}$

$\Rightarrow \arcsin\left(\dfrac{c-5}{6}\right) = \dfrac{\pi}{6} \quad \Rightarrow \quad \dfrac{c-5}{6} = \dfrac{1}{2} \quad \Rightarrow \quad c = 8$ *[8 marks]*