Revision: Algebra, functions and equations (Topics 1 & 2)

Coursebook chapters: 1–8; 15; 25

1. The function f is defined by $f(x) = \frac{1}{2}\ln(3x)$ for $x > 0$.

(a) State the range of f.

(b) Find an expression for $f^{-1}(x)$.

(c) Find the exact solution of the equation $f(x) = 5$.

(accessible to students on the path to grade 3 or 4) [5 marks]

2. Given that $z = a + b\text{i}$ and that $z^* + 2z = 3\text{i}$, find the values of a and b.

(accessible to students on the path to grade 3 or 4) [4 marks]

3. $f(x) = ax^3 - x^2 + 2x + b$

The remainder when $f(x)$ is divided by $(x + 2)$ is 5 and the remainder when $f(x)$ is divided by $(2x - 1)$ is 3.

Find the values of a and b.

(accessible to students on the path to grade 3 or 4) [5 marks]

4. Do not use a calculator to answer this question.

Two functions are defined by $f(x) = 3x^2$ and $g(x) = 4e^x + 1$.

(a) State the range of $g(x)$

(b) Solve the equation $f(g(x)) = 75$.

(accessible to students on the path to grade 3 or 4) [8 marks]

5. A student owns two Biology textbooks and six French textbooks.

(a) In how many ways can he select one Biology and three French textbooks?

(b) In how many ways can he arrange the books on the shelf so that the two biology textbooks are **not** next to each other?

(accessible to students on the path to grade 5 or 6) [6 marks]

6. A function is defined by $f(x) = x^2 - 3$ for $x \in \mathbb{R}$.

(a) Sketch the graph of $y = 2f(x - 1)$.

(b) Solve the inequality $|2f(x - 1)| < 5$.

(accessible to students on the path to grade 5 or 6) [5 marks]

7. Arun and Bea are starting training for a swimming race. On the first day they both swim 500 m . On each subsequent day, Arun swims 25 m more than the previous day, and Bea swims 5% farther than on the previous day.

(a) Find the total distance Arun will swim over the first 20 days.

(b) On which day will Bea first swim more than 1000 m?

(c) After how many days will Bea have swum a total of 5000 m more than Arun?

(accessible to students on the path to grade 5 or 6) [9 marks]

8. For the rational function $f(x) = \dfrac{x + a}{2x - b}$,

(a) State the equation of the vertical asymptote.

(b) In the case $a = 3$, $b = 5$, solve the equation $f(x) = f^{-1}(x)$.

(accessible to students on the path to grade 5 or 6) [5 marks]

9. (a) Show that $(x - 3)$ is a factor of $p(x) = 2x^3 - 5x^2 - 6x + 9$.

(b) Factorise $p(x)$ completely.

(c) Hence sketch the graph of $y = p(x)$.

(accessible to students on the path to grade 5 or 6) [9 marks]

10. Use mathematical induction to prove that $5^n + 9^n + 2$ is divisible by 4 for all $n \geq 1$.

(accessible to students on the path to grade 5 or 6) [9 marks]

11. Do not use a calculator for this question.

Find the general solution of the system of equations:

$$\begin{cases} x - 2y + z = 5 \\ 2x + y + 2z = 11 \\ 4x - 3y + 4z = 21 \end{cases}$$

(accessible to students on the path to grade 5 or 6) [7 marks]

12. (a) Use mathematical induction to prove that:

$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$ for $n \geq 1$.

(b) Let $1, \omega, \omega^2, \omega^3, \omega^4$ be the solutions of the equation $z^5 = 1$.

(i) Write ω in the form $re^{\mathrm{i}\theta}$.

(ii) Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$.

(accessible to students on the path to grade 5 or 6) [10 marks]

13. The two roots of the quadratic equation $ax^2 + bx + c = 0$ differ by 1. Prove that $b^2 - 4ac = a^2$.

(accessible to students on the path to grade 7) [6 marks]

14. The first three terms of the binomial expansion of $(x + p)^n$ are $x^n + 20x^{n-1} + 180x^{n-2}$.

Find the values of n and p.

(accessible to students on the path to grade 7) [6 marks]

15. When the polynomial $p(x)$ is divided by $(x^2 + 3x + 2)$ the remainder is $5x + 1$. Find the remainder when $p(x)$ is divided by $(x + 2)$.

(accessible to students on the path to grade 7) [4 marks]

16. The cubic equation $x^3 + bx^2 + cx + d = 0$, with $b > 0$, has roots α, β and γ such that $\alpha + \beta = \gamma$.

(a) Show that $\alpha + \beta = -\frac{b}{2}$ and $\alpha\beta = \frac{2d}{b}$.

(b) α and β are solutions of the quadratic equation $x^2 + mx + n = 0$. Express m and n in terms of b and d.

(c) Hence show that, if the equation $x^3 + bx^2 + cx + d = 0$ has three real roots, then $b^3 \geq 32d$.

(accessible to students on the path to grade 7) [9 marks]