Chapter notes: 15 Complex numbers

Overview

This chapter covers the whole complex numbers topic. Many books separate the 'algebraic' (Cartesian) and 'trigonometric' (polar) topics, but we feel that the connections between the two representations are essential. If you would like to cover the 'algebraic' part first, you can start with sections 15A, 15C (omitting Key point 15.7), 15D and 15E. We think that this chapter needs around 10 hours of teaching time.

Introductory problem

The introductory problem is an example of a question that can be answered more easily using complex numbers (although it can also be attempted using the compound angle identities). By the end of the chapter, students should understand that complex numbers may not 'exist', but they can still be used as a tool to solve 'real' problems. 'Theory of knowledge issues' and 'From another perspective' boxes on pages 477, 518 and 519, and Worked example 15.20, invite discussion of this issue. The worked solution is given at the end of the chapter, page 521; the idea being that students should be able to answer the question using the methods covered in the chapter.

An alternative introduction to complex numbers is provided in the document 'Complex numbers and the cubic equation'. This shows the original use of complex numbers in applying Cardano's formula for the cubic equation. We suggest working through it with the stronger students.

15A Definition and basic arithmetic of i, p476

The 'From another perspective' box should make students realise that this is not the first time they have been asked to accept a new 'type' of number. You could ask them to think what answer a seven-year old would give to the question: 'Can ten cakes be divided between three children?'.

15B Geometric interpretation, p482

The 'Research explorer' box mentions the Fundamental Theorem of Algebra, which states that every polynomial of degree n has exactly n (real or complex) roots (some of which may be repeated). We will meet this result in section 15D.

Students may want to research applications of complex numbers in geometry (these come about because multiplication by a complex number corresponds to a combination of rotation and enlargement in an Argand diagram). This then raises the question whether there is a three-dimensional equivalent of complex numbers that can help in some geometrical problems. One option is quaternions (a four-dimensional equivalent of complex numbers, but with properties for describing rotations in three dimensions), which are mentioned in the 'From another perspective' box on page 506. If you have already covered vectors you may want to discuss the similarities and differences between vectors of complex numbers and the type of geometrical properties they can describe.

Hints for grade 7 questions:

11. Write z and w in Cartesian form, and derive the relationship $\tan\theta_z = -\dfrac{1}{\tan\theta_w}$.

15C Properties of complex conjugates, p487

Hints for grade 7 questions:

17. ERRATA: The question should say 'either $|z| = 1$ or z is real'.
Use the fact that if a number is real then $z = z^*$.

19. Notice that the expression for the imaginary part can be simplified further to $\tan\frac{\theta}{2}$ by using double-angle identities.

15D Complex solutions to polynomial equations, p493

Hints for grade 7 questions:

10. (a) If the graph is tangent to the x-axis at $x = 5$, this root must be repeated an even number of times.

11. (b) Divide by the quadratic factor $(z - 2\text{i})(z + 2\text{i})$.

(c) Use the exam hint from page 495 to combine factors corresponding to conjugate roots.

15E Sums and products of roots of polynomials, p498

This section is new to the syllabus, which only explicitly mentions sums and products of roots rather than the rest of the Vieta's formulae (e.g. $x_1x_2 + x_2x_3 + x_3x_1 = \frac{c}{a}$ for a cubic). Nevertheless, we give a few examples of deriving and using them (see Worked example 15.18).

Hints for grade 7 questions:

9. Write the expressions for the sum and product of the roots.

10. (c) Use the identities derived in part (a).

15F Operations in polar form, p504

This section has links to compound-angle identities (see section 12B) and proof by induction (see chapter 25).

The 'Research explorer' box on page 507 mentions logarithms of complex numbers:
$\ln r \operatorname{cis} \theta = \ln r + \text{i}(\theta + 2k\pi)$. Note that this is a multi-valued function, which is a concept that the students will not have met, but you can compare it to the problem of $\pm$ with square roots, or the fact that a complex number has n nth roots.

Hints for grade 7 questions:

8. See Worked example 15.20.

15G Complex exponents, p510

Euler's form (including the name!) is included in the new syllabus. Complex exponents are not explicitly included, but questions using Euler's form to work out complex logarithms have appeared on examination papers before. Questions 5 and 8 look at trigonometric functions of complex numbers.

Hints for grade 7 questions:

8. Create a quadratic equation in e^{iz}.

15H Roots of complex numbers, p512

Hints for grade 7 questions:

7. (b) Pair up factors corresponding to conjugate roots.

8. (b) Use the formula for the sum of a geometric series.

9. Notice that $\omega^3 = 1$.

10. (c) Compare the equation to the expansion from part (b), then use the result from (a).

15I Using complex numbers to derive trigonometric identities, p517

This section links De Moivre's theorem with binomial expansion. If you have already covered integration of trigonometric functions, you can look at Worked example 19.8 and Question 11 in section 19C.

The 'Theory of knowledge issues' box on page 519 mentions several applications of complex numbers to derive 'real' results. The formula for the cubic equation is discussed in the document 'Complex numbers and the cubic equation'. In physics, complex numbers are often used to analyse periodic phenomena. This is because the quantities involved have both a magnitude and a phase, and complex numbers allow us to combine those two pieces of information into a single variable.

We can achieve a similar aim with vectors. However, the advantage of complex numbers is that they can also be divided and raised to a power. This allows for calculations which are more complicated. For example, a complex number $5e^{i(7t-1.5)}$ could represent a wave with amplitude 5, frequency 7 (i.e. wavelength $\frac{2\pi}{7}$) and initial phase 1.5. In electronics, impedance is a generalisation of resistance that includes both the normal resistance and the reactance (the phase lag between the current and the voltage in an alternating current circuit). It can be represented by a single complex number, $Z = R + iX$, where R is the resistance and X the reactance. If the alternating voltage and current are also described by complex numbers, the Ohm's law holds in the form $Z = \frac{V}{I}$.

ERRATA: The first equation at the top of page 518 should read $e^{i\theta} = \cos\theta + i\sin\theta$.

Hints for grade 7 questions:

4. This is similar to Worked example 15.27.

5. (c) Notice that both the numerator and the denominator tend to zero, so it is not possible to find the limit without using the result from part (b). You may want to link this idea for finding limits of the form $\frac{0}{0}$ to differentiation from first principles in section 16B, and in particular to the result that $\lim_{x \to 0} \frac{\sin x}{x} = 1$, which is explored on Fill-in proof sheet 16.