Complex numbers and the cubic equation

Preamble

In the course of the section, we will see how complex numbers arise from the formula for solving the cubic equation, practise doing complex arithmetic, and finally use it to find real solutions of some cubic equations.

We know that the quadratic equation $ax^2 + bx + c = 0$ can be solved by applying the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

A similar formula for the cubic equation was discovered by Cardano and Tartaglia in the 16th century. The formula is quite complicated. Here we will look at only the special case of cubic equations with no x^2 term.

The equation $x^3 + px + q = 0$ has solutions given by:

$$x = \frac{-p}{3u} + u$$

where $u = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$.

We know that the cubic equation can have up to three real solutions. Notice that the ± in front of the square root may give us two different solutions. In this sheet we shall see how this equation can actually give three solutions.

Research Explorer

Find out how the formula can be extended to deal with cubic equations which include the x^2 term. The quantity $\frac{q^2}{4} + \frac{p^3}{27}$ plays a role that is similar to the discriminant of a quadratic equation – investigate how it tells us how many real solutions the cubic equation has. There are other methods of solving - cubic equations – for example some involve using trigonometry and the compound-angle formula.

Let us see how the formula works on an example of the equation $x^3 + 9x - 26 = 0$.

Here $p = 9$, $q = -26$

So $u = \sqrt[3]{13 \pm \sqrt{196}} = -1$ or 3.

When $u = -1$, $x = \frac{-9}{-3} - 1 = 2$.

When $u = 3$, $x = \frac{-9}{9} + 3 = 2$.

So in this case, both possible values of u give the same solution for x.

Plotting the graph confirms that $x = 2$ is the only real solution of the equation.

Exercise

Apply the formula to solve the equation $x^3 + 6x - 20 = 0$.

Answer

$x = \sqrt[3]{10+\sqrt{108}} + \sqrt[3]{10-\sqrt{108}}$ which actually equals 2. (Both values of u give the same answer.)

Graphing shows that this is the only real root.

Let us now apply the formula to an equation which has three real roots: $x^3 - 3x = 0$. We can see by factorising that the roots are 0, $\sqrt{3}$ and $-\sqrt{3}$.

In the formula we take $p = -3$, $q = 0$.

Then, $u = \sqrt[3]{0 \pm \sqrt{0+\frac{-27}{27}}} = \sqrt[3]{\pm\sqrt{-1}}$.

Before complex numbers this result could not be solved – yet we know it should lead to three real solutions.

Using the methods from 15H we can see that there are three possible values for $\sqrt[3]{\mathrm{i}}$: $-\mathrm{i}$, $\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}$ and $-\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}$.

When $u = -\mathrm{i}$: $x = \frac{1}{-\mathrm{i}} + (-\mathrm{i}) = \mathrm{i} - \mathrm{i} = 0$.

When $u = \frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}$: $x = \frac{2}{\sqrt{3}+\mathrm{i}} + \left(\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\right) = \frac{2\left(\sqrt{3}-\mathrm{i}\right)}{3+1} + \left(\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}\mathrm{i} + \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i} = \sqrt{3}$.

When $u = -\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}$, a similar calculation gives the third solution $x = -\sqrt{3}$.

You should check that using the three possible values for $\sqrt[3]{-\mathrm{i}}$ gives the same three solutions.

Conclusion

The main purpose of this section was to show that although we may have doubts about the existence of imaginary numbers, they can be used as a tool to help find a real answer. The solution of the cubic equation is one example of such use. Others can be found in the work on differential equations, trigonometry and applications in electronics.