Self-assessment answers: 20 Further applications of calculus

1. $\pi\int_0^{\pi}\sin^2 3x\,\mathrm{d}x = \pi\int_0^{\pi}\frac{1}{2}(1-\cos 6x)\,\mathrm{d}x$

$=\frac{\pi}{2}\left[x=\frac{\sin 6x}{6}\right]_0^{\pi}$

$=\frac{\pi^2}{2}$ *[7 marks]*

2. $\frac{\mathrm{d}r}{\mathrm{d}t}=6\sqrt{t}$

$V=\frac{4}{3}\pi r^3 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}r}=4\pi r^2$

By chain rule, $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}r}\times\frac{\mathrm{d}r}{\mathrm{d}t}=24\pi r^2\sqrt{t}$

When $t = 5$ and $r = 26$, $\frac{\mathrm{d}V}{\mathrm{d}t}=113971\text{ cm}^3\text{ s}^{-1}$ *[5 marks]*

3. (a) $V=\pi r^2 h$, $A=2\pi r^2+2\pi rh$

$\Rightarrow\ A=\frac{2\pi r^2 h}{h}+\sqrt{4\pi^2 r^2 h^2}=\frac{2V}{h}+\sqrt{4V\pi h}=\frac{1264}{h}+\sqrt{2528\pi h}$ as required.

(b) Minimum surface area when $\frac{\mathrm{d}A}{\mathrm{d}h}=0$,

$\Rightarrow\ \frac{\mathrm{d}A}{\mathrm{d}h}=-1264h^{-2}+\sqrt{\frac{632\pi}{h}}=0$

$\Rightarrow\ h=\left(\frac{1264}{\sqrt{632\pi}}\right)^{\frac{2}{3}}=9.30\text{ cm}$

$\Rightarrow\ A=408\text{ cm}^2$ *[6 marks]*

4. (a) $a = \dfrac{dv}{dt} = 3e^{-2t}\cos t - 6e^{-2t}\sin t = 3e^{-2t}(\cos t - 2\sin t)$

When at maximum velocity, $a = 0 \Rightarrow \tan t = 0.5$ (on the first occasion, since the exponential means that it will never again reach that level).

$\Rightarrow t = 0.464$ s

(b) $a(3) = -0.00946 \text{ ms}^{-2}$

(c) v is positive for $0 \le t \le \pi$, so distance travelled equals displacement at $t = 3$.

distance $= \int_0^3 v\,\mathrm{d}t = 0.601$ m (GDC)

(d) Displacement $x(t) = \int_0^t 3e^{-2u}\sin u\,\mathrm{d}u$

$$= \left[-\frac{3}{2}e^{-2u}\sin u\right]_0^t + \int_0^t \frac{3}{2}e^{-2u}\cos u\,\mathrm{d}u \qquad \text{(integration by parts)}$$

$$= -\frac{3}{2}e^{-2t}\sin t + \left(\left[-\frac{3}{4}e^{-2u}\cos u\right]_0^t - \int_0^t \frac{3}{4}e^{-2u}\sin u\,\mathrm{d}u\right) \qquad \text{(integration by parts)}$$

$$= -\frac{3}{2}e^{-2t}\sin t + \left(\frac{3}{4} - \frac{3}{4}e^{-2t}\cos t - \frac{1}{4}x(t)\right)$$

$$\Rightarrow \quad x(t) = \frac{4}{5}\left(\frac{3}{4} - \frac{3}{4}e^{-2t}\cos t - \frac{3}{2}e^{-2t}\sin t\right) = \frac{3}{5}(1 - e^{-2t}\cos t - 2e^{-2t}\sin t)$$

[12 marks]

