Fill-in proof 1 Formula for choosing objects

We are going to demonstrate that…

The number of ways of choosing r objects out of n, $\binom{n}{r} = \dfrac{n!}{n!(n-r)!}$.

You need to know…

- there are $n!$ ways to permute n objects (Theorem 1)
- the product principle: $n(A \text{ AND } B) = n(A) \times n(B)$ (Theorem 2).

Proof

Fill in the missing expressions and reasons to explain what is being done in each line.

For some lines you may simply refer to Theorem 1 or 2.

	Approach	**Reasons**
1	Let's count the number of ways of arranging n objects in an unusual way: Select r of the objects (x ways) AND Permute those r objects (y ways) AND Permute the remaining objects (z ways) x is the quantity we are trying to find.	Setting up the proof
2	The total number of ways of doing this is	Theorem 1
3	But we can also think of it as $x \times y \times z$	Theorem
4	where $y = r!$	Theorem 1
5	and $z =$	Theorem
6	So we can form the equation $n! = x \times r! \times (n-r)!$	Theorem 2
7	Rearranging this gives $$x = \frac{n!}{r!(n-r)!}$$ Which completes the proof.	

Questions for reflection

1. Is the definition that $0! = 1$ consistent with the result for the number of ways of selecting no objects out of n?

2. This proof utilises what is called the *overcounting principle*. Can you see why it might be called that?

3. Try to construct a similar proof to show that the number of ways of arranging n objects in which there is a set of a identical objects, another set of b identical objects and a third set of c identical objects (as well as some other distinguishable objects) is $\frac{n!}{a!b!c!}$. Hence find the number of arrangements of the letters in the word MISSISSIPI.

4. Try to show that the number of ways of arranging n objects in a circle is $(n-1)!$

5. Prove that $\binom{n}{r} = \binom{n}{n-r}$.

   Can you interpret this result in terms of counting arguments?

6. Prove that $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.

   Can you interpret this result in terms of counting arguments?

Fill-in proof 2 Proving log rules

We are going to demonstrate that…

$\log_a xy = \log_a x + \log_a y$

You need to know…

- the definition of logarithms: $b = a^x \Rightarrow x = \log_a b$ (Theorem 1)
- the law of exponents stating $a^m \times a^n = a^{m+n}$ (Theorem 2)

Proof

Fill in the missing expressions and reasons to explain what is being done in each line. For some lines you may simply refer to Theorem 1 or 2.

	Approach	Reasons
1	Let $x = a^m$ and $y = a^n$ Then $m =$ ____ and $n =$ ____	Theorem 1
2	Let $z = xy$	
3	$\therefore z = a^m \times a^n = a^{m+n}$	Theorem ____
4	$\Rightarrow \log_a z = m + n$	Theorem ____
5	$\therefore \log_a(xy) = \log_a x + \log_a y$	Substituting from previous lines

Questions for reflection

1. At what point does this proof break down for $a = 1$?

2. Does this proof demonstrate that $\log(-1 \times -1) = 2\log(-1)$?

3. Give an example to show why the implication in the fourth line only goes in one direction.

3. Use the fact that $a^m \div a^n = a^{m-n}$ to prove that $\log_a \frac{x}{y} = \log_a x - \log_a y$.

4. Use the fact that $(a^m)^n = a^{m \times n}$ to prove that $\log_a x^p = p \log_a x$.

Fill-in proof 3 Proving the quadratic formula

We are going to demonstrate that…

If $ax^2+bx+c=0$ then $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

You need to know…

- how to complete the square.

Proof

Fill in the missing expressions and reasons to explain what is being done in each line.

		Approach	Reasons
1		$ax^2+bx+c=0$	
2	$\Rightarrow$	$x^2+\dfrac{b}{a}x+\dfrac{c}{a}=0$	$\boxed{\quad\quad}$
3	$\Leftrightarrow$	$\left(x+\dfrac{b}{2a}\right)^2-\boxed{\quad}+\dfrac{c}{a}=0$	Completing the square
4	$\Leftrightarrow$	$\left(x+\dfrac{b}{2a}\right)^2=\boxed{\quad}$	Isolating the square
5	$\Leftrightarrow$	$=\dfrac{\boxed{\quad}}{4a^2}$	Writing the RHS over a common denominator
6	$\Leftrightarrow$	$x+\dfrac{b}{2a}=\pm\sqrt{\boxed{\quad}}$	Undoing the squaring
7	$\Leftrightarrow$	$=\pm\dfrac{\sqrt{b^2-4ac}}{2a}$	Simplify the square root
8	$\Leftrightarrow$	$x=-\dfrac{b}{2a}\pm\dfrac{\sqrt{b^2-4ac}}{2a}$	Isolating x
		$=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$	

Questions for reflection

1. At what point does this proof break down when $a=0$?
2. At stage 6 are we saying that $\sqrt{4a^2}=2a$? If so, are we assuming that a is positive?
3. Can this method be extended to form a general solution to $ax^3+bx^2+cx+d=0$?

Fill-in proof 4 Arithmetic series and the story of Gauss

Carl Friedrich Gauss (1777–1855)

Carl Gauss was amongst the most eminent mathematicians of the 19th century. His many contributions included great strides in number theory, statistics and physics. He was a child prodigy and there is a famous legend about a lesson where his teacher was hoping to keep him quiet by asking him to add together all of the numbers from1 to 100. The teacher was somewhat disappointed when he replied with the correct answer within seconds. It is believed that he applied a procedure similar to the one used in this proof.

We are going to demonstrate that...

$$S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

You need to know...

- the definition of an arithmetic series
- that the nth term of an arithmetic series is $u_1 + (n-1)d$.

Proof

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	$S_n = \underbrace{[u_1]+[u_1+d]+\ldots+u_1+(n-2)d+[u_1+(n-1)d]}_{n\text{ terms}}$	Definition of S_n
2	$\Leftrightarrow\ S_n = \underbrace{[\qquad\quad]+[\qquad\quad]+\ldots+[u_1+d]+[u_1]}_{n\text{ terms}}$	Reversing the order of the sum does not change the answer
3	$\therefore\ 2S_n = \underbrace{[2u_1+(n-1)d]+[\qquad\quad]+\ldots+[\qquad\quad]}_{n\text{ terms}}$	Adding the first two expressions, combining terms in the square brackets
4	$\Leftrightarrow\ 2S_n = \quad\left(2u_1+(n-1)d\right)$	Collecting like terms
5	$\Leftrightarrow\ S_n = \frac{n}{2}\left(2u_1+(n-1)d\right)$	

Question for reflection

1. Can you use this method to find the sum of all the numbers from 1 to 100?

2. Use this formula to show that the sum of the first n odd numbers is n^2. Can you explain this result in any other ways? Try using diagrams. Which proof is 'best'?

Fill-in proof 5 Self-similarity and geometric series

We are going to demonstrate that…

$$S_n = \frac{u_1\left(r^n - 1\right)}{r-1}$$

You need to know…

- the definition of a geometric series
- that the nth term of a geometric series is $u_1 r^{n-1}$

Proof

Fill in the missing expressions and reasons to explain what is being done in each line.

		Approach		Reasons
1		$S_n = u_1 + u_1 r + u_1 r^2 + \ldots + u_1 r^{n-2} + u_1 r^{n-1}$	(1)	Definition of S_n
2	$\Leftrightarrow$	$rS_n = u_1 r + u_1 r^2 + u_1 r^3 + \ldots + u_1 r^{n-1} + u_1 r^n$	(2)	$\boxed{\qquad\qquad}$
3	$\therefore$	$rS_n - S_n = \boxed{\qquad}$		(2)-(1), eliminating identical terms
4	$\Leftrightarrow$	$S_n(\boxed{\quad}) = u_1(\boxed{\quad})$		Factorising both sides
5	$\Rightarrow$	$S_n = \dfrac{u_1\left(r^n - 1\right)}{r-1}$		Dividing by $r-1$

Questions for reflection

1. This proof does not work when $r = 1$.
   At which stage does it break down? What is the formula for S_n when $r = 1$?

2. Does this proof work when $r = -1$? Can you find a simplified version of the formula in this case?

3. This proof appeals to the very important mathematical idea of self-similarity – looking to get similar structures in two different ways, so that things cancel out. Use this idea to evaluate:

   (a) $\sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}$

   (b) $\dfrac{1}{1 + \dfrac{2}{1 + \dfrac{2}{1 + \ldots}}}$

Fill-in proof 6 Remainder theorem

We are going to demonstrate that…

If $\dfrac{f(x)}{x-a} \equiv q(x) + \dfrac{r}{x-a}$ where $q(x)$ is a polynomial, then $r = f(x)$.

You need to know…

- the definition of a polynomial
- the meaning of the word 'identity'.

Proof

Fill in the missing expressions and reasons to explain what is being done in each line.

		Approach	Reasons
1		$\dfrac{f(x)}{x-a} \equiv q(x) + \dfrac{r}{x-a}$	Definition of an algebraic remainder
2	$\Rightarrow$	$f(x) \equiv (x-a)q(x) + \boxed{\quad}$	Multiplying both sides by $x-a$
3	$\Leftrightarrow$	$f(a) = (\boxed{\quad} - a)q(\boxed{\quad}) + r$	Choosing a useful value of x
4	$\Leftrightarrow$	$f(a) = \boxed{\quad}$	

Questions for reflection

1. Where in the proof is it important that $q(x)$ is a polynomial? Does $f(x)$ have to be a polynomial?

2. Why did we need to multiply by $(x-a)$ at stage 2 rather than substitute in $x = a$ in stage 1?

> This highlights an issue with this proof. To get around this the remainder is defined using
>
> $$f(x) \equiv (x-a)q(x) + r.$$

3. Adapt the proof above to find the remainder when $f(x)$ is divided by $(bx-a)$.

4. The factor theorem states that if $f(a) = 0$ then $f(x)$ has a factor of $(x-a)$. Show that this follows directly from the remainder theorem.

Fill-in proof 7 Linking the binomial expansion and counting principles

We are going to demonstrate that...

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}a^1b^{n-1} + \binom{n}{n}b^n$$

You need to know...

- how to manually expand brackets
- that the number of ways of choosing r objects out of n is $\binom{n}{r}$.

Proof

Fill in the missing expressions to explain what is being done in each line.

1 Think about what happens when you expand a set of brackets:

$$\underset{1}{(a+b)}\underset{2}{(a+b)}\underset{3}{(a+b)}\ldots\underset{n}{(a+b)}$$

We shall focus on the case when $n=4$ to show the structure of the proof.

2 One way of thinking about the expansion is that it will be the sum of every possible product formed by picking one element from each bracket and multiplying them together. There is only one way of getting a^4:

$$(\underline{a}+b)(\underline{a}+b)(\underline{a}+b)(\underline{a}+b)$$

There are four different ways to get a^3b:

$$(\underline{a}+b)(\underline{a}+b)(\underline{a}+b)(a+\underline{b})$$

Each of these ways contribute an a^3b term so in the final expansion there will be a^3b.

3 This coefficient occurred because we can pick 3 brackets to take an 'a' from (and then be left with one to take a 'b' from) in different ways.

4 In general, if we are expanding $(a+b)^n$ to get the term in a^rb we need to pick r brackets to get 'a' from. This can be done in different ways, so in the final expansion there will be

5 We can do this for any integer value of r from to n, and all of these terms add together to give the required result.

This 'proof' is far less algebraic than many proofs you will have seen. Is it a proof or an explanation? What is the difference between proof and explanation in mathematics?

Questions for reflection

1. Where in the proof is it important that n is a positive integer?

2. Can you extend this proof to prove a formula for the trinomial expansion $(a+b+c)^n$?

3. By considering a geometric series, can you find an infinitely long expansion for $(1-x)^n$? Does it always work?

Fill-in proof 8 Cosine rule

We are going to demonstrate that…

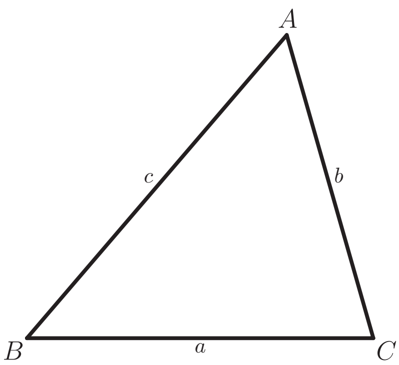
$$c^2 = a^2 + b^2 - 2ab\cos C$$

You need to know…

- the use of cosine in right-angled triangles: $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ (Theorem 1)
- the use of sine in right-angled triangles: $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ (Theorem 2)
- Pythagoras' theorem in a right-angled triangle: $c^2 = a^2 + b^2$ (Theorem 3)
- the Pythagorean identity $\sin^2\theta + \cos^2\theta \equiv 1$. (Theorem 4)

Proof

Fill in the missing expressions and reasons to explain what is being done in each line. You may refer to Theorem 1, 2, 3 or 4.

	Approach	Reasons
1		Drawing a diagram representing a general triangle
2	Draw AD perpendicular to BC	Split the diagram into right-angled triangles
3	In triangle ADC we can express the remaining two sides in terms of b and C: $CD =$	Theorem 1
4	$AD =$	Theorem 2

5	In triangle *BDA* we can now deduce that	
	$BD =$ ______	Total length is of *BC* is *a*
6	But *BDA* is a right-angled triangle so	
	$c^2 = (b\sin C)^2 +$ ______	Theorem 3
7	$= b^2 \sin^2 C +$ ______	Expanding brackets
8	$= a^2 + b^2 - 2ab\cos C$	Theorem 4

Questions for reflection

1. What assumptions have been made in drawing the triangle in stage one? Does this proof apply only to this particular triangle? Are there any other cases that need to be considered?

2. Derive similar proofs for these other cases to show that the cosine rule applies for all triangles.

Fill-in proof 9 Double angle cosine formula

We are going to demonstrate that…

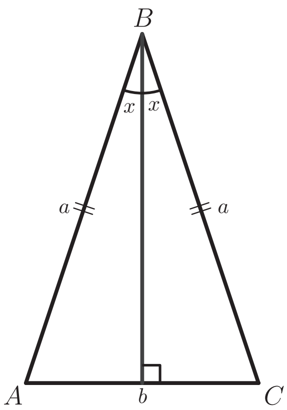
$$\cos 2x = 1 - 2\sin^2 x$$

You need to know…

- the use of sine in right-angled triangles: $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ (Theorem 1)
- the cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$ (Theorem 2)

Proof

Fill in the missing expressions and reasons to explain what is being done in each line. You may refer to Theorem 1 or 2.

	Approach	Reasons
1		Drawing a diagram with two identical isosceles right-angled triangles
2	Considering just one right-angled triangle we can write the base side in terms of *a*: ______	Theorem ___
3	$\Leftrightarrow b = 2a\sin x$ (1)	______
4	Considering the large triangle again we can write the base in terms of *a*: $b^2 = a^2 + a^2 - 2a \times a \times \cos 2x$	Theorem ___
5	$= 2a^2($ ______ $)$	Factorising
6	$\Leftrightarrow ($ ______ $)^2 = 2a^2($ ______ $)$	______
7	$\Leftrightarrow$ ______ $= 2a^2($ ______ $)$	Expanding bracket
8	$\therefore$ ______ $=$ ______	We can divide by $2a^2$ since $a \neq 0$
9	$\cos 2x = 1 - 2\sin^2 x$	Rearranging

Questions for reflection

1. What assumptions have been made in drawing the triangle in stage one? What type of angles does this proof apply to?

2. Use the identity $\sin^2 x + \cos^2 x \equiv 1$ to deduce the identities:

$$\cos 2x = 2\cos^2 x - 1$$

and

$$\cos 2x = \cos^2 x - \sin^2 x$$

Fill-in proof 10 Sine compound angle formula

We are going to demonstrate that…

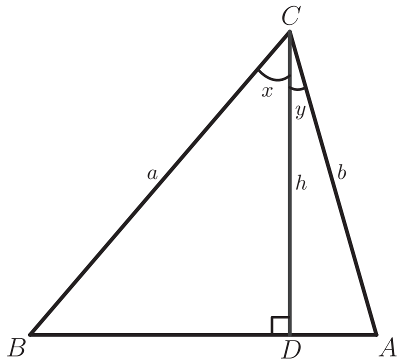
$$\sin(x+y)=\cos y\sin x+\cos x\sin y$$

You need to know…

- the use of cosine in right-angled triangles: $\cos\theta=\dfrac{\text{adjacent}}{\text{hypotenuse}}$ (Theorem 1)
- the area of a triangle: area $=\dfrac{1}{2}ab\sin C$. (Theorem 2)

Proof

Fill in the missing expressions and reasons to explain what is being done in each line. You may refer to Theorem 1 or 2.

	Approach	Reasons
1		Drawing a diagram with two right-angled triangles with the same height.
2	Considering the left right-angled triangle we can write the height in terms of a and x: $h=a\cos x$	Theorem 1
3	By symmetry we can write h in terms of b and y: $h=b\cos y$	
4	The area of the left-hand triangle is: $\frac{1}{2}a\times(b\cos y)\times\sin x$	Theorem 2
5	The area of the right-hand triangle is: $\frac{1}{2}b\times(a\cos x)\times\sin y$	Theorem 2
6	The area of the whole triangle is: ______	Theorem 2
7	But the total area is the sum of the two smaller triangles: ______ = ______ + ______	
8	$\Rightarrow\sin(x+y)=\cos y\sin x+\cos x\sin y$ This is the required result.	______

Questions for reflection

1. For any pair of acute angles x and y can we always draw two right-angled triangles with the same height?

2. Is the argument 'by symmetry' in stage 3 valid?

3. Given that $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$, use a substitution $y = -z$ to show that $\sin(x - z) = \sin x \cos z - \cos x \sin z$.

Fill-in proof 11 Cosine compound angle formula

We are going to demonstrate that...

$\cos(x+y)=\cos x\cos y-\sin x\sin y$

You need to know...

- the compound angle sine identity:

$$\sin(x-y)=\cos y\sin x-\cos x\sin y \qquad \text{(Theorem 1)}$$

- the complementary angle identities:

$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta \qquad \text{(Theorem 2)}$$

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta \qquad \text{(Theorem 3)}$$

- a simple algebraic identity:

$$A-(B+C)=(A-B)-C \qquad \text{(Theorem 4)}$$

Proof

Fill in the missing expressions and reasons to explain what is being done in each line. You may refer to Theorem 1, 2, 3 or 4.

	Approach	Reasons
1	$\cos(x+y)=\sin\left(\frac{\pi}{2}-(x+y)\right)$	Theorem
2	$=\sin\left(\left(\frac{\pi}{2}-x\right)-y\right)$	Theorem
3	$=\cos y\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)\sin y$	Theorem
4	$=\cos y\cos x-\cos\left(\frac{\pi}{2}-x\right)\sin y$	Theorem
5	$=\cos y\cos x-\sin x\sin y$	Theorem

Questions for reflection

1. Given that $\cos(-\theta)=\cos\theta$ and $\sin(-\theta)=-\sin\theta$, use a substitution $y=-z$ to show that $\cos(x-z)=\cos x\cos z+\sin x\sin z$.

2. Does setting $x=y$ provide a proof that $\cos 2x=\cos^2 x-\sin^2 x$?

3. Does this proof apply to obtuse angles? Reflex angles?

Fill-in proof 12 Deriving the scalar product

We are going to demonstrate that...

$$\cos\theta = \frac{(a_1b_1 + a_2b_2 + a_3b_3)}{|a||b|}$$

You need to know...

- the geometric interpretation of vectors
- the component form of vectors
- the cosine rule:

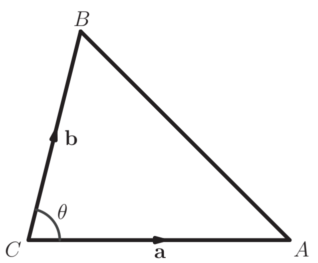
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \qquad \text{(Theorem 1)}$$

- the modulus of a vector in component form:

$$\left|\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right| = \sqrt{a_1^2 + a_2^2 + a_3^2} \qquad \text{(Theorem 2)}$$

Proof

Fill in the missing expressions and reasons to explain what is being done in each line. You may refer to Theorem 1 or 2.

	Approach	Reasons
1	 $\overrightarrow{CA} = a, \overrightarrow{CB} = b$, so $\overrightarrow{AB} = \boxed{}$	Geometric property of vectors
2	If $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then $b - a = \begin{pmatrix} \boxed{} \end{pmatrix}$.	Component property of vectors
3	In this triangle we have $\cos\theta = \dfrac{\lvert a\rvert^2 + \lvert b\rvert^2 - \boxed{}}{2\boxed{}}$	Theorem $\boxed{}$
4	$\lvert a\rvert^2 = \boxed{}$ $\lvert b\rvert^2 = \boxed{}$ $\lvert b - a\rvert^2 = \boxed{}$	Theorem $\boxed{}$

5	$\lvert a\rvert^2+\lvert b\rvert^2-\lvert b-a\rvert^2=$ $a_1^2+a_2^2+a_3^2+b_1^2+b_2^2+b_3^2-$	Algebraic simplification
6	$\cos\theta=$ $=\dfrac{(a_1b_1+a_2b_2+a_3b_3)}{\lvert a\rvert\lvert b\rvert}$	Substituting into expression for $\cos\theta$

Questions for reflection

1. Which would you say is the definition of the scalar product:

(a) $a.b=\lvert a\rvert\lvert b\rvert\cos\theta$

(b) $\dfrac{a.b}{\lvert a\rvert\lvert b\rvert}=\cos\theta$

(c) $a.b=a_1b_1+a_2b_2+a_3b_3$?

Is mathematics about defining something and then investigating its properties or defining something because it has a use?

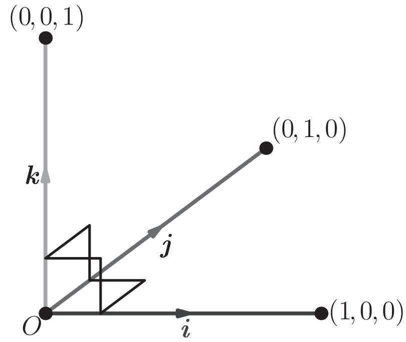
Fill-in proof 13 Deriving the vector product

We are going to demonstrate that…

$$\boldsymbol{a}\times\boldsymbol{b}=\begin{pmatrix} a_2b_3-a_3b_2 \\ a_3b_1-a_1b_3 \\ a_1b_2-a_2b_1 \end{pmatrix}$$

You need to know…

- the component form of vectors
- the definition of the vector product: $\boldsymbol{a}\times\boldsymbol{b}=|a||b|\sin\theta\hat{\boldsymbol{n}}$ where $\hat{\boldsymbol{n}}$ is a unit vector in the direction perpendicular to both a and b and in a sense given by the right hand rule.
- the system of basis vectors ***i***, ***j*** and ***k***:



- The following properties of the vector product:

$$\boldsymbol{a}\times\boldsymbol{b}=-\boldsymbol{b}\times\boldsymbol{a} \qquad \text{(Theorem 1)}$$

$$\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c})=\boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{b}\times\boldsymbol{c} \qquad \text{(Theorem 2)}$$

$$\boldsymbol{a}\times\boldsymbol{a}=0 \qquad \text{(Theorem 3)}$$

Theorem 1 follows from the definition of the direction of the vector product.

Theorem 3 is true because $\sin(0)=0$.

The proof of Theorem 2 requires further knowledge of vectors, so we will accept it without proof.

Proof

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	Write ***a*** and ***b*** in terms of the unit base vectors ***i, j, k***: $\boldsymbol{a}=a_1\boldsymbol{i}+a_2\boldsymbol{j}+a_3\boldsymbol{k}$ $\boldsymbol{b}=b_1\mathbf{i}+b_2\boldsymbol{j}+b_3\boldsymbol{k}$	
2	$a\times b=a_1\boldsymbol{i}(b_1\boldsymbol{i}+b_2\boldsymbol{j}+b_3\boldsymbol{k})$ $+a_2\boldsymbol{j}(b_1\boldsymbol{i}+b_2\boldsymbol{j}+b_3\boldsymbol{k})$ $+a_3\boldsymbol{k}(b_1\boldsymbol{i}+b_2\boldsymbol{j}+b_3\boldsymbol{k})$	Theorem
3	$=(a_1b_1)(\boldsymbol{i}\times\boldsymbol{i})+(a_1b_2)(\boldsymbol{i}\times\boldsymbol{j})+(\quad\boldsymbol{i}\times\mathbf{k})$ $+(\quad\boldsymbol{j}\times\boldsymbol{i})+(\quad\boldsymbol{j}\times\boldsymbol{j})+(a_2b_3)(\boldsymbol{j}\times\boldsymbol{k})$ $+(\quad\boldsymbol{k}\times\boldsymbol{i})+(\quad\boldsymbol{k}\times\boldsymbol{j})+(\quad\boldsymbol{k}\times\boldsymbol{k})$	Theorem 2
4	$\boldsymbol{i}\times\boldsymbol{i}=0$	Theorem

5 $\boldsymbol{i} \times \boldsymbol{j} = |\boldsymbol{i}||\boldsymbol{j}|\sin\frac{\pi}{2} \times \boldsymbol{k}$ — Definition of vector product

$= \boldsymbol{k}$

6 $\therefore \boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k}$ — Theorem 1

7 Completing the table below in a similar fashion gives:

			Second	
		$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
First	$\boldsymbol{i}$	**0**	$\boldsymbol{k}$	$\boldsymbol{j}$
	$\boldsymbol{j}$	$-\boldsymbol{k}$	**0**	$\boldsymbol{i}$
	$\boldsymbol{k}$	$\boldsymbol{j}$	$\boldsymbol{i}$	**0**

Therefore — Using results from table

$a \times b = (a_1 b_2)\boldsymbol{k} + (a_1 b_3)(-\boldsymbol{j}) + \quad +$

$\quad + \quad +$

$= \quad \boldsymbol{i} + \quad \boldsymbol{j} + \quad \boldsymbol{k}$ — Factorising

Fill-in proof 14 Solutions to real polynomials

We are going to demonstrate that...

If z is a solution of a polynomial with real coefficients then z^* is also a solution.

You need to know...

- $(zw)^* = z^* w^*$ (Theorem 1)
- $(z+w)^* = z^* + w^*$ (Theorem 2)
- $(z^n)^* = (z^*)^n$ (Theorem 3)

Proof

Fill in the missing expressions and reasons to explain what is being done in each line.

		Approach	Reasons
1		We start off by defining a polynomial function, $f(x)$, of order n with real coefficients. The coefficient of x^r is a_r, so $f(x) = a_n x^n + a_{n-1}x^{n-1} \ldots + a_1 x + a_0$. If z is a solution to $f(x) = 0$ then $a_n z^n + a_{n-1}z^{n-1} \ldots + a_1 z + a_0 = 0$. We need to establish from this that $f(z^*) = 0$.	Setting up the problem algebraically
2	$\Leftrightarrow$	$(a_n z^n + a_{n-1}z^{n-1} \ldots + a_1 z + a_0)^* = 0^*$	Taking complex conjugates of both sides
3	$\Leftrightarrow$	$(a_n z^n + a_{n-1}z^{n-1} \ldots + a_1 z + a_0)^* = 0$	Since 0 is real $0^* = 0$
4	$\Leftrightarrow$	$(a_n z^n)^* + (a_{n-1}z^{n-1})^* \ldots + (a_1 z)^* + a_0^* = 0$	Theorem
5	$\Leftrightarrow$	$a_n^* (z^n)^* + (a_{n-1}^*)(z^{n-1})^* \ldots + a_1^* z^* + a_0^* = 0$	Theorem
6	$\Leftrightarrow$	$a_n (z^n)^* + (a_{n-1})(z^{n-1})^* \ldots + a_1 z^* + a_0 = 0$	
7	$\Leftrightarrow$	$a_n (z^*)^n + a_{n-1}(z^*)^{n-1} + \ldots a_1 z^* + a_0 = 0$	Theorem
8	$\Leftrightarrow$	$f(z^*) = 0$	

Questions for reflection

1. Which stage means that this proof is only valid for polynomials with real coefficients?
2. Could this proof be extended to rational functions? How about any function?
3. Does this proof mean that there will always be an even number of solutions to a real polynomial?

Fill-in proof 15 Differentiating polynomials

We are going to demonstrate that...

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^n\right)=nx^{n-1} \text{ and that } \frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)+h(x)\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)\right)+\frac{\mathrm{d}}{\mathrm{d}x}\left(h(x)\right)$$

You need to know...

- differentiation from first principles: $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ (Theorem 1)
- the binomial expansion: $(a+b)^n=\sum\limits_{r=0}^{n}\dbinom{n}{r}a^r b^{n-r}$ (Theorem 2)
- the formula for binomial coefficient: $\dbinom{n}{r}=\dfrac{n!}{r!(n-r)!}$ (Theorem 3)

Proof 1

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	If $f(x)=x^n$ $f'(x)=\lim\limits_{h\to 0}\dfrac{(x+h)^n-x^n}{h}$	Theorem $\boxed{}$
2	$=\lim\limits_{h\to 0}\dfrac{x^n+\boxed{}+\binom{n}{2}x^{n-2}h^2+\sum_{r=3}^{n}\binom{n}{r}h^r x^{n-r}-x^n}{h}$	Theorem $\boxed{}$
3	$=\lim\limits_{h\to 0}\dfrac{\boxed{}+\binom{n}{2}x^{n-2}h^2+\sum_{r=3}^{n}\binom{n}{r}h^r x^{n-r}}{h}$	$\boxed{}$
4	$=\lim\limits_{h\to 0}\dfrac{\boxed{}+\frac{n(n-1)}{2}x^{n-2}h^2+\sum_{r=3}^{n}\binom{n}{r}h^r x^{n-r}}{h}$	Theorem $\boxed{}$
5	$=\lim\limits_{h\to 0}\left(\boxed{}+\dfrac{n(n-1)}{2}x^{n-2}h+\sum\limits_{r=3}^{n}\dbinom{n}{r}h^{r-1}x^{n-r}\right)$	$\boxed{}$
6	$=nx^{n-1}$	Taking the limit

Questions for reflection

1. Why could we not take the limit until stage 6?

2. Why do all the terms represented by the sigma notation go to zero in the limit?

3. For what values of n is this proof valid? At what stage in the proof is this restriction required?

4. You may want to extend the proof to show that it is also true for negative n (when you've learnt the product rule) and rational n (when you've learnt implicit differentiation). Real n will need some university-level mathematics!

Proof 2

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	If $f(x)=g(x)+h(x)$ $f'(x)=\lim\limits_{h\to 0}\dfrac{g(x+h)+h(x+h)-(\quad\quad)}{h}$	Theorem
2	$=\lim\limits_{h\to 0}\dfrac{(g(x+h)-g(x))+(\quad\quad)}{h}$	Rearranging
3	$=\lim\limits_{h\to 0}\dfrac{g(x+h)-g(x)}{h}+\lim\limits_{h\to 0}\quad\quad$	Separating the fractions
4	$=g'(x)+h'(x)$	Theorem

Questions for reflection

1. Can you combine the results to prove that $\dfrac{\mathrm{d}}{\mathrm{d}x}(ax^n)=anx^{n-1}$?

2. Can you further prove that all polynomials can be differentiated in this fashion?

Fill-in proof 16 Small angle approximations

We are going to demonstrate that...

$$\sin\theta \approx \theta$$

when θ is small and in radians.

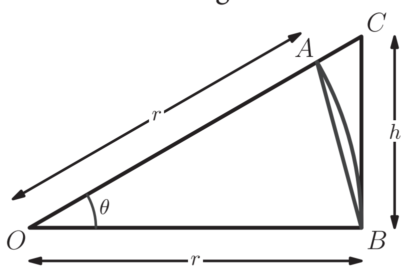
You need to know...

- the formula for the area of a triangle: $\frac{1}{2}\text{base} \times \text{height}$ (Theorem 1)
- the formula for the area of a triangle: $\frac{1}{2}ab\sin C$ (Theorem 2)
- the formula for the area of a sector: $\frac{1}{2}r^2\theta$ (Theorem 3)
- $\cos\theta \approx 1$ when θ is very small (Theorem 4)
- in a right-angled triangle $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$ (Theorem 5)
- $\tan\theta = \frac{\sin\theta}{\cos\theta}$ (Theorem 6)

Proof

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	Consider this diagram.	



	Approach	Reasons
2	The area of the triangle $OAB =$	Theorem 2
3	The height $h =$	Theorem 5
4	The area of the triangle $OCB =$	Theorem 1
5	The area of the sector $OAB =$	Theorem 3
6	From the diagram in terms of areas: triangle $OAB <$ sector $OAB <$ triangle OCB	
7	θ θ θ	

8	$\Rightarrow \quad \sin\theta < \theta < \tan\theta$	
9	$\Rightarrow \quad 1 < \dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta}$	Dividing by $\sin\theta$ and using Theorem
10	In the limit as θ gets very small $\cos\theta \approx 1$ $1 < \dfrac{\theta}{\sin\theta} <$	Theorem 4
11	If $\dfrac{\theta}{\sin\theta}$ is squeezed between 1 and 1 then $\theta \approx \sin\theta$	

Questions for reflection

1. Does the radius of the circle drawn in stage 1 affect the proof?

2. Is the statement made in stage 6 true for all possible diagrams? What about all values of θ?

3. At which stage in the proof is it important that θ is in radians? Find a small angle approximation for $\sin\theta$ when θ is in degrees. You may want to use this result to find the derivative of $\sin x$ when x is in degrees.

4. Why is it justified to say that $\cos\theta \approx 1$ but we do not use the approximation $\sin\theta \approx 0$? Use the small angle sine approximation to find an improved small angle approximation to $\cos\theta$ up to a quadratic term.

Fill-in proof 17 Differentiating trigonometric functions

We are going to demonstrate that…

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos(x)$$

You need to know…

- differentiation from first principles: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ (Theorem 1)
- small angle approximations: $\sin x \approx x$ and $\cos x \approx 1$ (Theorem 2)
- compound angle formula: $\sin(A+B) = \sin A \cos B + \sin B \cos A$ (Theorem 3)

Proof

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \lim_{h \to 0} \frac{\sin \square - \sin x}{h}$	Theorem $\square$
2	$= \lim_{h \to 0} \frac{\sin x \cos h \, \square - \sin x}{h}$	Theorem $\square$
3	$= \lim_{h \to 0} \frac{\square \sin x + \square \times \cos x - \sin x}{h}$	Theorem 2
4	$= \lim_{h \to 0} \frac{\square}{h}$	Cancelling sin x
5	$= \lim_{h \to 0} \cos x$	$\square$
6	$= \cos x$	Taking the limit

Questions for reflection

1. Apply the above method to prove that $\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$.
2. Where in the proof is it important that x is in radians?
3. $\sin x$ is only approximately equal to x, so why are we justified in using an exact equals sign in stage 3?

Fill-in proof 18 Differentiating logarithmic functions graphically

We are going to demonstrate that...

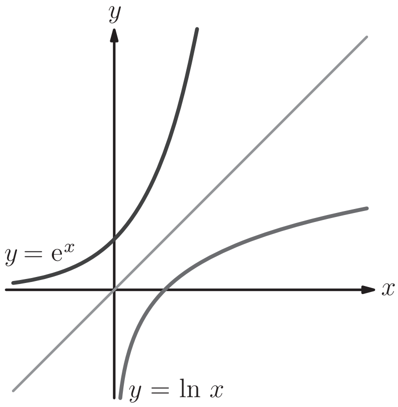
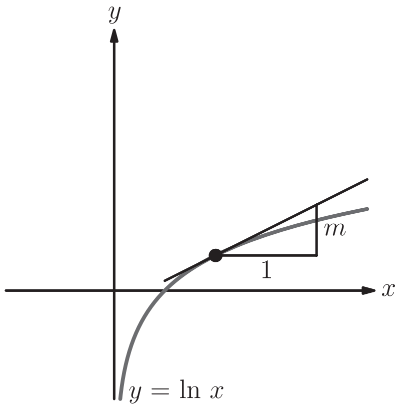
$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}$$

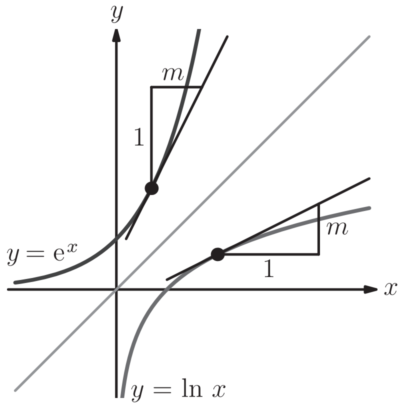
You need to know...

- $\ln x$ and e^x are inverse functions (Theorem 1)
- that the graphs of inverse functions are reflections in the line $y = x$ (Theorem 2)
- the gradient of any point on the graph of $y = \mathrm{e}^x$ is the same as the y-coordinate (Theorem 3)
- the gradient of a straight line is $\dfrac{\text{change in } y}{\text{change in } x}$ (Theorem 4)

Proof

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	Consider the graphs of $y = \mathrm{e}^x$ and $y = \ln x$ 	
2	Focus on the point $(x, \ln x)$. We shall call the gradient at this point m. We can draw a triangle with part of the tangent making explicit that the gradient is m. 	Theorem

3	We can then look at the reflection of this curve in the line $y = x$. It produces the curve $y = e^x$ 	Theorem and Theorem
4	The gradient is the same as the y-coordinate, which is	Theorem 3
5	But the gradient in terms of m is	Theorem 4
6	$\therefore m = \frac{1}{x}$	Taking the limit

Questions for reflection

1. Apply the above method to prove that $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$.

2. When you learn a technique called implicit differentiation you will be able to prove this in a significantly easier way. When you meet implicit differentiation try it.

Fill-in proof 19 Integration from first principles

We are going to demonstrate that…

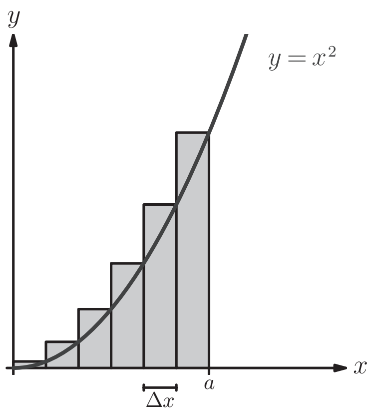
The area under the graph of $y = x^2$ from 0 to a is $\frac{a^3}{3}$.

You need to know…

- $\sum_{1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ (Theorem 1)

Proof

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	The area can be approximated by splitting it into lots of rectangles each with width Δx: 	
2	There will be of these rectangles	
3	The rth such rectangle will have the right edge of the rectangle at $x = r\Delta x$ and will have height $y = r^2(\Delta x)^2$	
4	Therefore it will have area	Area of a rectangle
5	The total area will be approximately: $\sum_{r=1}^{r=} r^2(\Delta x)^3$	
6	$=(\Delta x)^3 \sum_{r=1}^{r=\frac{a}{\Delta x}} r^2$	Since $(\Delta x)^3$ is a constant over the sum
7	$=\frac{(\Delta x)^3\left(\frac{a}{\Delta x}\right)()()}{6}$	Theorem 1
8	$=\frac{(\Delta x)^3\left(\frac{a}{\Delta x}\right)\left(\frac{a+\Delta x}{\Delta x}\right)()}{6}$	Writing each bracket over a single denominator

9	$=\dfrac{a(a+\Delta x)(2a+\Delta x)}{6}$	Simplifying
10	The approximation becomes exact as Δx gets very small: $\text{area} = \lim\limits_{\Delta x \to 0} \dfrac{a(a+\Delta x)(2a+\Delta x)}{6}$ $=\dfrac{a(a+0)(2a+0)}{6}$ $=\dfrac{a^3}{3}$	

Questions for reflection

1. We can approximate the area using rectangles that lie strictly below the curve. Does this produce the same result?

2. Use this proof to explain why the notation for this integration is $\int_0^a y\,\mathrm{d}x$.

3. (Hard) By considering lots of squares of dimensions Δx by Δy show that the area can also be thought of as

$$\sum_{i=1}^{i=\frac{a}{\Delta x}} \left(\sum_{j=1}^{j=\frac{y}{\Delta y}} \Delta y \right) \Delta x.$$

Finding an area by integration is really finding a double integral:

$$\int_{x=0}^{x=a} \int_{y=0}^{y=f(x)} \mathrm{d}A$$

Double and triple integrals have many uses in physics and engineering. For example, have a look at how they can be used to find the centre of mass of a non-uniform shape.

Fill-in proof 20 Fundamental theorem of calculus

We are going to demonstrate that…

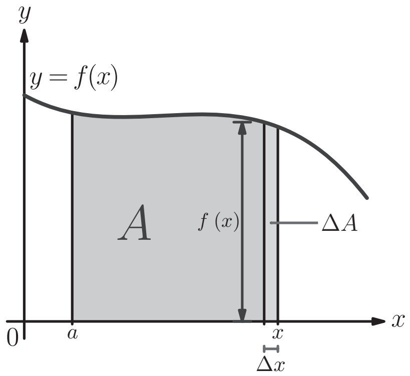
The process of finding the area under a graph is the opposite of differentiation.

You need to know…

- $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$ (Theorem 1)

Proof

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	Consider the area under the graph $y = f(x)$ between a and x. By changing the upper limit, x, we will change the area: 	
2	If the change in x is very small, the new area can be approximated by a rectangle with area: $\Delta A \approx$	
3	$\Leftrightarrow \dfrac{\Delta A}{\Delta x} \approx$	
4	In the limit, $f(x) = \lim\limits_{\Delta x \to 0} \dfrac{\Delta A}{\Delta x} = \dfrac{\mathrm{d}A}{\mathrm{d}x}$.	
5	Interpreting this: $f(x)$ is the derivative of A with respect to x so to find A we must find a function that differentiates to give $f(x)$.	

Questions for reflection

1. The volume of a sphere is $V = \dfrac{4}{3}\pi r^3$.

Find $\dfrac{\mathrm{d}V}{\mathrm{d}r}$ and see if you can find out what the resulting formula gives. Can you interpret this geometrically?

2. How would we have to adapt the proof above if $f(x) < 0$?

Fill-in proof 21 Product rule

We are going to demonstrate that...

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

You need to know...

- $\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ (Theorem 1)
- the convention that Δa means a small change in a.

Proof

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	Let $y = uv$ where u, v are functions of the independent variable x. If x changes by a small amount (Δx) our new independent variable is going to be $x + \Delta x$. But u, v and y will also be affected.	Setting up the problem
2	u will become $u + \Delta u$ v will become $v + \boxed{}$ y will become $y + \Delta y$	
3	But we can also use the link between y, u and v: $y + \Delta y = (u + \Delta u)\boxed{}$	
4	$= uv + \boxed{}$	Expanding brackets
5	$\therefore \Delta y = \boxed{}$	Since $y = uv$
6	$\Rightarrow \frac{\Delta y}{\Delta x} = \boxed{}$	Dividing both sides by Δx
7	Taking the limit: $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left(\boxed{}\right)$ $= v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + \lim_{\Delta x \to 0} \Delta u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}$	
8	$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = v\frac{\mathrm{d}u}{\mathrm{d}x} + \boxed{\phantom{u\frac{dv}{dx}}} + \boxed{} \times \frac{\mathrm{d}v}{\mathrm{d}x}$ $= v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}$	Theorem $\boxed{}$

Questions for reflection

1. In this proof did we assume that Δx is positive?

2. In stage 7 have we assumed that we can take the limit of each part of a sum and a product separately?

3. If $\frac{\mathrm{d}}{\mathrm{d}x}(x)=1$ use the product rule to prove that $\frac{\mathrm{d}}{\mathrm{d}x}(x^2)=2x$.

4. Criticise the following 'alternative' proof:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(x^2) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\underbrace{x+x+\ldots+x+x}_{x\text{ times}}\right)\\
&= \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}(x)+\frac{\mathrm{d}}{\mathrm{d}x}(x)+\ldots+\frac{\mathrm{d}}{\mathrm{d}x}(x)+\frac{\mathrm{d}}{\mathrm{d}x}(x)}_{x\text{ times}}\\
&= \underbrace{1+1+\ldots+1+1}_{x\text{ times}}\\
&= x
\end{aligned}$$

Fill-in proof 22 Quotient rule

We are going to demonstrate that…

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right)=\frac{v\frac{\mathrm{d}u}{\mathrm{d}x}-u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

You need to know…

- the product rule: $\frac{\mathrm{d}}{\mathrm{d}x}(uv)=u\frac{\mathrm{d}v}{\mathrm{d}x}+v\frac{\mathrm{d}u}{\mathrm{d}x}$ (Theorem 1)
- the chain rule: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}$ (Theorem 2)
- the rule of exponents: $v^{-n}=\frac{1}{v^n}$ (Theorem 3)
- $\frac{\mathrm{d}}{\mathrm{d}x}(x^n)=nx^{n-1}$ (Theorem 4)

Proof

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right)=\frac{\mathrm{d}}{\mathrm{d}x}(u\times v^{-1})$	Rewrite the quotient as a $\square$
2	$=v^{-1}\frac{\mathrm{d}u}{\mathrm{d}x}+\square$	Theorem 1
3	$=v^{-1}\frac{\mathrm{d}u}{\mathrm{d}x}+u\frac{\square}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}v}(v^{-1})$	Theorem 2
4	$=v^{-1}\frac{\mathrm{d}u}{\mathrm{d}x}+u\frac{\mathrm{d}v}{\mathrm{d}x}\square$	Theorem $\square$
5	$=\square\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{\square}{v^2}\frac{\mathrm{d}v}{\mathrm{d}x}$	Theorem $\square$
6	$=\frac{\square}{v^2}\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{\square}{v^2}\frac{\mathrm{d}v}{\mathrm{d}x}$	Create a common denominator of v^2
7	$=\frac{v\frac{\mathrm{d}u}{\mathrm{d}x}-u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$	

Question for reflection

1. Is the quotient rule necessary, or can all problems be solved using just the product rule?

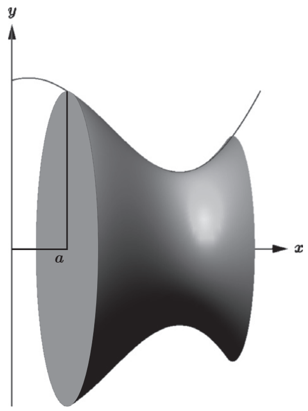
2. Is the quotient rule useful?

Fill-in proof 23 Volumes of revolution

We are going to demonstrate that…

The volume of a revolution when a curve is rotated fully around the x-axis is given by:

$$V = \int_a^b \pi y^2 \, dx$$

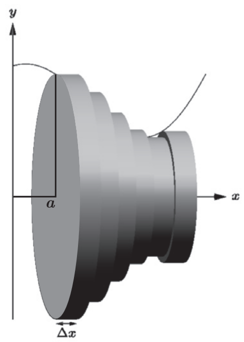


You need to know…

- that an integral is the limit of a sum of very small terms (Theorem 1)
- that the volume of a disc is $\pi r^2 h$. (Theorem 2)

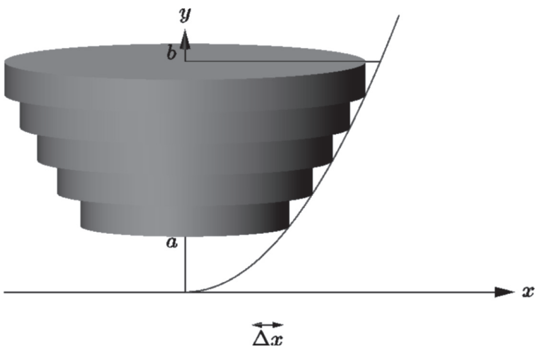
Proof

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	The volume can be split up into discs: 	
2	The radius of each disc is the y-coordinate and the width is Δx, therefore the volume is ______	Theorem ___
3	The total volume is approximately: $V \approx \sum_a^b$ ______	
4	As we make the discs smaller the volume gets more and more accurate: $V = \lim_{\Delta x \to 0} \sum_a^b$ ______ $= \int_a^b \pi y^2 \, dx$	Theorem ___

Questions for reflection

1. Use a similar argument to show that the volume of revolution of a curve rotated around the y-axis is $V = \int_a^b \pi x^2 \, dy$.



2. (Hard) By considering the volume being split into a series of thin cylinders show that the volume of revolution around the x-axis can be given by $V = \int_a^b 2\pi |x||y| \, dx$.

Fill-in proof 24 An alternative formula for variance

We are going to demonstrate that…

The variance is $\overline{x^2} - \overline{x}^2$.

You need to know…

- the definition of variance: $\sigma^2 = \dfrac{\sum_i (x_i - \overline{x})^2}{n}$ (Theorem 1)
- the definition of mean $y = \dfrac{\sum_i y_i}{n}$ (Theorem 2)
- $\sum_{i=1}^{i=n} 1 = n$ (Theorem 3)
- if a factor inside a sum is constant across the sum it can become a factor of the whole sum: (Theorem 4)

  $$\sum_i f(i)g(j) = g(j)\sum_i f(i)$$

- a finite sum of a sum can be split into two sums: (Theorem 5)

  $$\sum_i f(i) + g(i) = \left(\sum_i f(i)\right) + \left(\sum_i g(i)\right)$$

Proof

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	$\sigma^2 = \dfrac{\sum_i (x_i - \overline{x})^2}{n}$	Theorem 1
2	$= \dfrac{\sum_i x_i^2 - 2x_i\overline{x} + \overline{x}^2}{n}$	Expanding brackets
3	$= \dfrac{\sum_i x_i^2}{n} - \dfrac{\sum_i 2x_i\overline{x}}{n} + \dfrac{\sum_i \overline{x}^2}{n}$	Theorem
4	$= \dfrac{\sum_i x_i^2}{n} - 2\overline{x}\dfrac{\sum_i x_i}{n} + \overline{x}^2\dfrac{\sum_i 1}{n}$	Theorem
5	$= \overline{x^2} - 2 \times \overline{x} \times \overline{x} + \overline{x}^2$	Theorem and Theorem
6	$= \overline{x^2} - \overline{x}^2$	

Fill-in proof 25 Expectation and variance of the binomial distribution

We are going to demonstrate that...

If $X \sim B(n, p)$ then $\mathrm{E}(X) = np$ and $\mathrm{Var}(X) = np(1-p)$

You need to know......

- binomial expansion
- sigma notation
- how to calculate expectation and variance
- algebra of factorials and binomial coefficients.

Proof 1

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	If $X \sim B(n, p)$	
	$\mathrm{E}(X) = \sum_{r=0}^{r=n} rP(X=r)$	By the definition of expectation
	$= \sum_{r=0}^{r=n} r\binom{n}{r} p^r q^{n-r}$	
2	$\mathrm{E}(X) = \sum_{r=0}^{r=n} r\frac{n!}{r!(n-r)!} p^r q^{n-r}$	Using definition of $\binom{n}{r}$
3	$= 0 + \sum_{r=1}^{r=n} r\frac{n!}{r!(n-r)!} p^r q^{n-r}$	Splitting the sum into the first term and the rest.
4	$= \sum_{r=1}^{r=n} \frac{n!}{(r-1)!(n-r)!} p^r q^{n-r}$	Simplifying the factorials
5	$= np\sum_{r=1}^{r=n} \frac{(n-1)!}{(r-1)!(n-r)!} p^{r-1} q^{n-r}$	Taking out a factor of np
6	$= np\sum_{r=1}^{r=n} \boxed{\quad\quad} q^{n-r}$	Rewriting the factorial form as a binomial coefficient
7	$= np\sum_{a=0}^{a=b} \binom{b}{a} p^a q^{b-a}$	Substituting $r = \boxed{\quad}$ $\boxed{\quad}$
8	$= np(p+q)^b$	Recognising this as a $\boxed{\quad\quad\quad}$
	$= np(1)^b$ $= np$	Using the fact that $p + q = \boxed{\quad}$

Questions for reflection

1. Why did the sum have to be split up into two parts before the factorials were simplified?

2. Why were factors of n and p taken out in line 5?

Proof 2

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	$\mathrm{E}(X^2)=\sum_{r=0}^{r=n} r^2 \binom{n}{r} p^r q^{n-r}$	First we need to find $\mathrm{E}(X^2)$
2	$=\sum_{r=0}^{r=n} r^2 \left(\square\right) p^r q^{n-r}$	Rewriting the binomial coefficient in factorial form
3	$=0+\sum_{r=1}^{r=n} r\left(\dfrac{n!}{(\square)!(n-r)!}\right) p^r q^{n-r}$	Splitting the sum into the first term and the rest
4	$=\sum_{r=1}^{r=n}\left((r-1)+1\right)\left(\dfrac{n!}{(r-1)!(n-r)!}\right) p^r q^{n-r}$	Rewriting r as $(r-1)+1$
5	$=\sum_{r=1}^{r=n}(r-1)\left(\dfrac{n!}{(r-1)!(n-r)!}\right) p^r q^{n-r}+\sum_{r=1}^{r=n}\dfrac{n!}{(r-1)!(n-r)!} p^r q^{n-r}$	Splitting the sum into two sums
6	$=0+\sum_{r=2}^{r=n}\dfrac{n!}{(r-2)!(n-r)!} p^r q^{n-r}+\sum_{r=1}^{r=n}\dfrac{n!}{(r-1)!(n-r)!} p^r q^{n-r}$	Splitting the first sum into the first term and the rest
7	$=\sum_{r=2}^{r=n}\dfrac{n!}{(r-2)!(n-r)!} p^r q^{n-r}+\square$	Recognising the second sum from Proof 1
8	$=n(n-1)p^2\sum_{r=2}^{r=n}\dfrac{\square}{(r-2)!(n-r)!} p^{r-2} q^{n-r}+\square$	Taking out a factor of $n(n-1)p^2$
9	$=n(n-1)p^2\sum_{r=2}^{r=n}\square\; p^{\square} q^{n-r}+\square$	Rewriting the factorial form as a binomial coefficient
10	$=n(n-1)p^2\sum_{a=0}^{a=b}\binom{b}{a} p^b q^{b-a}+\square$	Substituting $r=\square$ $n=\square$
11	$=n(n-1)p^2(\square)^b+\square$	Recognising this as a binomial $\square$
12	$=n^2p^2-np^2+np$	Using the fact that $p+q=\square$
13	$\mathrm{Var}(X)=E(X^2)-[E(X)]^2$ $=n^2p^2-np^2+np-(\square)^2$	Using the definition of variance
14	$=np-np^2$ $=np(1-p)$	

Questions for reflection

1. Why was it useful to write r as $(r-1)+1$ in the 4th line?

2. Was the substitution in the 10th line necessary or just a useful way to simplify the algebra?

Fill-in proof 26 Deriving formulae for the Poisson distribution

We are going to demonstrate that...

If X is a variable following a Poisson distribution with mean m then $\mathrm{P}(X=x)=\frac{m^x e^{-m}}{x!}$.
If $\mathrm{P}(X=x)=\frac{m^x e^{-m}}{x!}$ then $\mathrm{E}(X)=m$ and $\mathrm{Var}(X)=m$.
The sum of two Poisson variables also follows a Poisson distribution.

Things you need to know...

- the binomial expansion
- the concept of a limit
- asymptotic definition of e^x: $\mathrm{e}^x=\lim\limits_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ (Theorem 1)
- series expansion of e^x: $\mathrm{e}^x=1+\frac{x}{1!}+\frac{x^2}{2!}\ldots$ (Theorem 2)

Proof 1: Deriving the distribution

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	When waiting for a bus you could consider each minute to be a binomial trial, or each second, or each millisecond. A Poisson distribution can be thought of as the limit of a binomial distribution as $n\to\infty$ and $p\to 0$ while the average, $m=np$, remains fixed.	
2	If $X\sim B(n,p)$ then $P(X=x)=$	
3	$=\frac{n!}{x!(n-x)!}\left(\frac{m}{n}\right)^x\left(1-\frac{m}{n}\right)^{n-x}$	Since $p=$
4	$=\frac{\overbrace{n(n-1)\ldots(n-x+1)}^{x\text{ terms}}}{x!}\left(\frac{m^x}{n^x}\right)\left(1-\frac{m}{n}\right)^n\left(1-\frac{m}{n}\right)^{-x}$	Simplifying the factorials
5	$\approx\frac{n^x}{x!}\left(\frac{m^x}{n^x}\right)\left(1-\frac{m}{n}\right)^n\left(1-\frac{m}{n}\right)^{-x}$	If n is much larger than k, $n-k\approx$
6	$=\frac{m^x}{x!}\left(1-\frac{m}{n}\right)^n\left(1-\frac{m}{n}\right)^{-x}$	Simplifying
7	$=\frac{m^x\mathrm{e}^{-m}}{x!}\left(1-\frac{m}{n}\right)^{-x}$	Theorem 1 applies since $n\to\infty$
8	$\approx\frac{m^x\mathrm{e}^{-m}}{x!}$ In the limit, this approximation is exact.	$1-\frac{m}{n}=1-p\approx$

Proof 2: Deriving the expectation

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	$\mathrm{E}(X)=\sum_{x=0}^{x=\infty} x\dfrac{m^x \mathrm{e}^{-m}}{x!}$	From the definition of expectation
2	$=0+\sum_{x=1}^{x=\infty} x\dfrac{m^x \mathrm{e}^{-m}}{x!}$	Splitting the sum into the first term and the rest
3	$=\mathrm{e}^{-m}\sum_{x=1}^{x=\infty} \dfrac{m^x}{(x-1)!}$	e^{-m} is a constant as far as the sum is concerned $\dfrac{x}{x!}=$
4	$=m\mathrm{e}^{-m}\sum_{x=1}^{x=\infty} \dfrac{m^{x-1}}{(x-1)!}$	Taking out a factor of m
5	$=m\mathrm{e}^{-m}\sum_{y=0}^{y=\infty} \dfrac{m^y}{y!}$	Substitute $y=$
6	$=m\mathrm{e}^{-m}\mathrm{e}^{m}=m$	Theorem

Questions for reflection

1. Why was it necessary to split the sum into the first term and the rest before simplifying the factorials?

2. Why was a substitution necessary?

Proof 3: Deriving the variance

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	$\mathrm{E}(X^2)=\sum_{x=0}^{x=\infty} x^2\dfrac{m^x \mathrm{e}^{-m}}{x!}$	First we need to find $\mathrm{E}(X^2)$
2	$=0+\sum_{x=1}^{x=\infty} x^2\dfrac{m^x \mathrm{e}^{-m}}{x!}$	Splitting the sum into the first term and the rest
3	$=\sum_{x=1}^{x=\infty} x\dfrac{m^x \mathrm{e}^{-m}}{(x-1)!}$	Simplifying the factorial
4	$=m\sum_{x=1}^{x=\infty} x\dfrac{m^{x-1} \mathrm{e}^{-m}}{(x-1)!}$	Taking out a factor of m
5	$=m\left(\sum_{x=1}^{x=\infty}(x-1)\dfrac{m^{x-1}\mathrm{e}^{-m}}{(x-1)!}+\sum_{x=1}^{x=\infty}\dfrac{m^{x-1}\mathrm{e}^{-m}}{(x-1)!}\right)$	Rewriting x as $(x-1)+1$
6	$=m\left(\sum_{y=0}^{y=\infty} y\dfrac{m^y \mathrm{e}^{-m}}{y!}+\sum_{y=0}^{y=\infty}\dfrac{m^y \mathrm{e}^{-m}}{y!}\right)$	Substituting
7	$=m\left(m+\sum_{y=0}^{y=\infty}\dfrac{m^y \mathrm{e}^{-m}}{y!}\right)$	First term in the bracket is the expectation which we found in Proof 2

8	$= m(m+1)$	Second term in the bracket is the total probability
9	$\therefore \mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ $= m^2 + m - m^2 = m$	Definition of variance

Question for reflection

1. Why was writing x as $(x-1)+1$ necessary?

Proof 4: Sum of two Poisson variables

If $X \sim Po(\lambda)$ and $Y \sim Po(\mu)$ where X and Y are independent then what is the distribution of $Z = X + Y$?

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	Consider all the different ways in which Z can take the value z. If $X=0$ then $Y=z$. If $X=1$ then $Y=z-1$, etc.	
2	$\therefore \mathrm{P}(Z=z) = \mathrm{P}(X=0)\mathrm{P}(Y=z)$ $+\mathrm{P}(X=1)\mathrm{P}(Y=z-1)\ldots$ $+\mathrm{P}(X=z)\mathrm{P}(Y=0)$	
3	$= \sum_{r=0}^{r=z}$ ______	Rewriting in sigma notation
4	$= \sum_{r=0}^{r=z} \frac{\lambda^r e^{-\lambda}}{r!} \times$ ______	Using the formula for the Poisson distribution
5	$= e^{-\lambda}e^{-\mu} \sum_{r=0}^{r=z} \frac{\lambda^r}{r!} \times \frac{\mu^{z-r}}{(z-r)!}$	Factors of $e^{-\lambda}$ and $e^{-\mu}$ can be taken out of the sum since they are constants
6	$= \frac{e^{-(\lambda+\mu)}}{z!} \sum_{r=0}^{r=z} \frac{z!}{r!(z-r)!} \lambda^r \mu^{z-r}$	We are close to having a binomial coefficient. Multiply by $z!$ in the sum to get to this, but then you have to divide by $z!$ too
7	$= \frac{e^{-(\lambda+\mu)}}{z!} \sum_{r=0}^{r=z}$ ______ $\lambda^r \mu^{z-r}$	Replace the factorials with a binomial coefficient
8	$= \frac{e^{-(\lambda+\mu)} (\lambda+\mu)^z}{z!}$ This is a Poisson distribution with mean $\lambda+\mu$.	We can recognise the sum as a ______

Questions for reflection

1. At which stage in this proof did you make the link with the binomial expansion? How would you look out for this in future?

2. At what stage in the proof did we use the fact that X and Y are independent?

Fill-in proof 27 Expectation algebra of linear expressions

We are going to demonstrate that...

$$\mathrm{E}(aX+b)=a\mathrm{E}(X)+b$$

$$\mathrm{Var}(aX+b)=a^2\mathrm{Var}(X)$$

You need to know...

- $\mathrm{E}(X)=\sum x_i p_i$ (Theorem 1)
- $\mathrm{E}(X^2)=\sum x_i^2 p_i$ (Theorem 2)
- $\mathrm{Var}(X)=\mathrm{E}(X^2)-\mathrm{E}(X)^2$ (Theorem 3)
- $\sum p_i = 1$ (Theorem 4)

Proof 1

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	$\mathrm{E}(aX+b)=\sum_i (ax_i+b)p_i$	Theorem 1
2	$=\sum_i ax_i p_i + \sum_i bp_i$	Separating the sum into two parts
3	$=a\sum_i x_i p_i + b\sum_i p_i$	Taking out constant factors from the sums
4	$=a\mathrm{E}(X)+b\sum_i p_i$	Theorem
5	$=a\mathrm{E}(X)+b$	Theorem

Proof 2

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	$\mathrm{E}\left([aX+b]^2\right)=\sum_i (ax_i+b)^2\, p_i$	Theorem 2
	$=\sum_i (a^2x_i^2 p_i + 2abx_i p_i + b^2 p_i)$	Multiplying out
2	$=\sum_i a^2x_i^2 p_i + \sum_i 2abx_i p_i + \sum_i b^2 p_i$	Separating the sum into three parts
3	$=a^2\sum_i x_i^2 p_i + 2ab\sum_i x_i p_i + b^2\sum_i p_i$	Taking out constant factors from the sums
4	$=a^2\mathrm{E}(X^2)+2ab\sum_i x_i p_i + b^2\sum_i p_i$	Theorem
5	$=a^2\mathrm{E}(X^2)+2ab\mathrm{E}(X)+b^2\sum_i p_i$	Theorem

6	$= a^2\mathrm{E}\left(X^2\right) + 2ab\mathrm{E}(X) + b^2$	Theorem
7	$\therefore \mathrm{Var}(aX+b) = a^2\mathrm{E}\left(X^2\right) + 2ab\mathrm{E}(X) + b^2 - \left[a\mathrm{E}(X)+b\right]^2$	Theorem
8	$= a^2\mathrm{E}\left(X^2\right) + 2ab\mathrm{E}(X) + b^2$ $-[\quad]$	Multiplying out brackets
9	$= a^2\mathrm{E}\left(X^2\right) -$	Simplifying
10	$= a^2\left(\mathrm{E}\left(X^2\right) - \mathrm{E}(X)^2\right)$	
11	$= a^2\mathrm{Var}(X)$	Theorem

Fill-in proof 28 Expectation of a sum of independent variables

We are going to demonstrate that…

$\mathrm{E}(X+Y)=\mathrm{E}(X)+\mathrm{E}(Y)$ if X and Y are independent variables.

You need to know…

- $\mathrm{E}(X)=\sum_i x_i p_i$ (Theorem 1)
- $\mathrm{E}\big(g(X,Y)\big)=\sum_i\sum_i g(x_i,y_i)\,\mathrm{P}(X=x_i\cap Y=y_i)$ (Theorem 2)
- $\mathrm{P}\big(X=x\cap Y=y\big)=\mathrm{P}(X=x)\,\mathrm{P}(Y=y)$ if X and Y are independent (Theorem 3)
- $\sum p_i=1$ (Theorem 4)
- a finite double sum of a sum can be split into two double sums (Theorem 5)

$$\sum_i\sum_j f(i,j)+g(i,j)=\left(\sum_i\sum_j f(i,j)\right)+\left(\sum_i\sum_j g(i,j)\right)$$

Several of these theorems may be unfamiliar and may themselves need to be proven.

Proof

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	$\mathrm{E}(X+Y)=\sum_i\sum_j(x_i+y_j)\,\mathrm{P}(X=x_i\cap Y=y_j)$	Theorem
2	$=\sum_i\sum_j(x_i+y_j)\mathrm{P}(X=x_i)\mathrm{P}(Y=y_j)$	Theorem
3	$=\sum_i\sum_j x_i\mathrm{P}(X=x_i)\mathrm{P}(Y=y_j)+\sum_i\sum_j y_j\mathrm{P}(X=x_i)\mathrm{P}(Y=y_j)$	Theorem
4	$=\sum_i\left(x_i\mathrm{P}(X=x_i)\sum_j\mathrm{P}(Y=y_j)\right)$ $+\sum_i\left(\mathrm{P}(X=x_i)\sum_j y_j\,\mathrm{P}(Y=y_j)\right)$	Properties of sums
5	$=\sum_i x_i\mathrm{P}(X=x_i)\times 1+\sum_i\mathrm{P}(X=x_i)\times\mathrm{E}(Y)$	Theorem and Theorem
6	$=\mathrm{E}(X)+\mathrm{E}(Y)\sum_i P(X=x_i)$	Theorem
7	$=\mathrm{E}(X)+\mathrm{E}(Y)$	Theorem

Questions for reflection

1. Prove that $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$ if X and Y are independent.

2. Prove that

$$\mathrm{E}\left(\left[X+Y\right]^2\right) = \mathrm{E}\left(X^2\right) + \mathrm{E}\left(Y^2\right) + \mathrm{E}(X)\mathrm{E}(Y)$$

   Hence prove that

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

3. The result has been proved for *independent* variables but it is actually true for all *uncorrelated* variables. Research the difference between independent and uncorrelated.

4. (Hard!) Prove that $\mathrm{Var}(XY) = \left[\mathrm{E}(X)\right]^2 \mathrm{Var}(Y) + \left[\mathrm{E}(\mathrm{Y})\right]^2 \mathrm{Var}(\mathrm{X}) + \mathrm{Var}(\mathrm{X})\mathrm{Var}(\mathrm{Y})$.

5. (Hard!) Investigate the rules of double sums for infinite sums.

Fill-in proof 29 Expectation and variance of the geometric distribution

We are going to demonstrate that…

If $X \sim DU(n)$ then $\mathrm{E}(X) = \dfrac{n+1}{2}$, $\mathrm{Var}(X) = \dfrac{n^2-1}{12}$

You need to know…

- the sum to infinity for a geometric series: $S_\infty = \dfrac{u_1}{1-r}$ if $|r|<1$
- calculus – especially the quotient rule

Lemma 1: Showing that $\displaystyle\sum_{x=1}^{x=\infty} xr^x = \frac{r}{(1-r)^2}$

Note: A *lemma* is a stepping stone towards a mathematical proof.

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	$\displaystyle\sum_{x=1}^{x=\infty} xr^x = \square\sum_{x=0}^{x=\infty} xr^{x-1}$	Property of sums
2	$\displaystyle = r\sum_{x=1}^{x=\infty} \frac{\mathrm{d}}{\mathrm{d}r}(\square)$	Linking the expression to a standard derivative
3	$\displaystyle = r\frac{\mathrm{d}}{\mathrm{d}r}\left(\sum_{x=1}^{x=\infty} r^x\right)$	The sum of a derivative is the derivative of the sum
4	$\displaystyle = r\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\square}{1-r}\right)$ if $\lvert r\rvert<1$	Sum of a geometric series
5	$\displaystyle = r\frac{(1-r)+r}{(1-r)^2}$	Applying the quotient rule
6	$\displaystyle = \frac{r}{(1-r)^2}$	Simplifying

Proof 1: Deriving the expectation

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	$\mathrm{E}(X)=\sum_{x=1}^{x=\infty} xpq^{x-1}$	Definition of expectation
2	$=\boxed{}\sum_{i=1}^{i=\infty} xq^{x}$	Using property of sums to link to Lemma 1
3	$=\dfrac{pq}{q(1-q)^2}$	Applying Lemma 1
4	$=\dfrac{p}{(1-q)^2}$	$\boxed{\phantom{\text{reason here}}}$
5	$=\dfrac{p}{p^2}$	Since $1-q=\boxed{}$
6	$=\dfrac{1}{p}$	

Lemma 2: Summing the series $\sum_{x=1}^{x=\infty} x(x-1)r^x$

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	$\sum_{x=1}^{x=\infty} x(x-1)r^x = r^2\sum_{x=0}^{x=\infty} x(x-1)r^{x-2}$	Property of sums
2	$=r^2\sum_{x=1}^{x=\infty}\boxed{}$	Linking the expression to a standard derivative
3	$=\boxed{}$	The sum of a derivative is the derivative of a sum
4	$=r^2\dfrac{\mathrm{d}^2}{\mathrm{d}r^2}\left(\dfrac{r}{1-r}\right)$ if $\lvert r\rvert<1$	Sum of a geometric series
5	$=\dfrac{2r^2}{(1-r)^3}$	Applying the quotient rule

Proof 2: Deriving the variance

Fill in the missing expressions and reasons to explain what is being done in each line.

	Approach	Reasons
1	$\mathrm{E}\left(X^2\right)=\sum_{x=1}^{x=\infty} x^2 pq^{x-1}$	Definition of expectation
2	$=\sum_{x=1}^{x=\infty}(x^2-x)pq^{x-1}+\sum_{x=1}^{x=\infty}xpq^{x-1}$	Rewriting x^2 as $(x^2-x)+x$
	$\square\, x(x-1)q^x+\sum_{x=1}^{x=\infty}xpq^{x-1}$	Using the rules of sums to link to Lemma 2
3	$=\frac{p}{q}\sum_{x=1}^{x=\infty}x(x-1)q^x+\mathrm{E}(X)$	Recognising the definition of expectation
4	$=\frac{p}{q}\frac{\square}{(1-q)^3}+\frac{1}{p}$	From Lemma 2 and Proof 1
5	$=\frac{2pq}{p^3}+\frac{1}{p}$ $=\frac{2q}{p^2}+\frac{1}{p}$	Since $1-q=p$
6	$\mathrm{Var}(X)=\mathrm{E}\left(X^2\right)-\left[\mathrm{E}(X)\right]^2$	Definition of variance
7	$=\frac{2q}{p^2}+\frac{1}{p}-\square$	From above
8	$=\frac{2q}{p^2}+\frac{p}{p^2}-\frac{1}{p^2}$	Using a common denominator
9	$=\frac{q+q+p-1}{p^2}$	
10	$=\frac{q}{p^2}$	Since $q+p=\square$