Self-assessment answers: 14 Lines and planes in space

1. (a) Direction vector is $\begin{pmatrix}1\\-1\\2\end{pmatrix}$, so the vector equation of l_1 is $\boldsymbol{r}=\begin{pmatrix}4\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\-2\end{pmatrix}$ (or equivalent).

In Cartesian form, this is $x-4=-y=\dfrac{3-z}{2}$.

(b) Intersection where $\begin{pmatrix}4\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\-2\end{pmatrix}=\begin{pmatrix}-4\\5\\-5\end{pmatrix}+t\begin{pmatrix}2\\-1\\2\end{pmatrix}$

$$\Rightarrow \begin{cases}4+\lambda=-4+2t & (1)\\ -\lambda=5-t & (2)\\ 3-2\lambda=-5+2t & (3)\end{cases}$$

$(1)+(2) \;\Rightarrow\; 4=1+t$

$\Rightarrow\; t=3$

$(2) \;\Rightarrow\; \lambda=-2$

$(3) \;\Rightarrow\; 3+4=-5+6$: False

These are skew lines and do not intersect. *[8 marks]*

2. (a) Normal vectors are $\boldsymbol{n}_1=\begin{pmatrix}4\\1\\-2\end{pmatrix}$ and $\boldsymbol{n}_2=\begin{pmatrix}1\\-1\\0\end{pmatrix}$.

Angle between planes is the same as angle between normals.

$\boldsymbol{n}_1\cdot\boldsymbol{n}_2=|\boldsymbol{n}_1||\boldsymbol{n}_2|\cos\theta$

$\Rightarrow\; \theta=\arccos\left(\dfrac{\boldsymbol{n}_1\cdot\boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}\right)$

$=\arccos\left(\dfrac{3}{\sqrt{21\times2}}\right)$

$=1.09$ radians ($62.4°$) (3SF)

(b) Direction vector $\boldsymbol{d}$ of intersecting line is perpendicular to both normals:

$$\boldsymbol{d} = \boldsymbol{n}_2 \times \boldsymbol{n}_1 = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$$

A point on both lines (by inspection) is (2, 0, 1).

So the vector equation of the intersecting line is $\boldsymbol{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$ *[8 marks]*

3. (a) If $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix}$,

$$\Rightarrow \begin{cases} 3 - t = 5 & (1) \\ 1 + t = -1 & (2) \\ 1 + 2t = -3 & (3) \end{cases}$$

(1) $\Rightarrow$ $= -2$

(2) $\Rightarrow$ $1 + t = -1$ which is consistent

(3) $\Rightarrow$ $1 + 2t = -3$ which is consistent

So (5, −1, −3) is on l.

(b) $\overrightarrow{AD} = \begin{pmatrix} 0 \\ 4 \\ 7 \end{pmatrix}$

$$\Rightarrow \overrightarrow{AD} \times \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -7 \\ 4 \end{pmatrix}$$

(c) Normal to the plane $\boldsymbol{n} = \begin{pmatrix} 1 \\ -7 \\ 4 \end{pmatrix}$, plane passes through (5, −1, −3).

So the Cartesian equation of the plane is $x - 7y + 4z = 0$.

(d) P lies on l, so P = $\begin{pmatrix}3\\1\\1\end{pmatrix}+t\begin{pmatrix}-1\\1\\2\end{pmatrix}$ for some t. Then $\overrightarrow{DP}=\begin{pmatrix}3-t-5\\1+t-3\\1+2t-4\end{pmatrix}=\begin{pmatrix}-t-2\\t-2\\2t-3\end{pmatrix}$.

Require DP perpendicular to $\begin{pmatrix}-1\\1\\2\end{pmatrix}$, so $\begin{pmatrix}-t-2\\t-2\\2t-3\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\2\end{pmatrix}=0$

$\Rightarrow$ $(t + 2) + (t - 2) + (4t - 6) = 0$

$\Rightarrow$ $t = 1$

$\Rightarrow$ P = (2, 2, 3) *[14 marks]*

